INTRODUCTION TO ALGEBRAIC GEOMETRY

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ABSTRACT. The materials are based upon essentially Griffiths-Harris and Beauville's book "Complex Algebraic Surfaces".

1. COMPLEX MANIFOLDS

Let W be an open subset of $\mathbb{C}^n$, $z_1, \dots, z_n$ be the coordinates for $\mathbb{C}^n$. We write

$$z_i = x_i + \sqrt{-1} y_i, \quad \frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}).$$

Let $f(z_1, \dots, z_n)$ be a continuous function on W. A well-known fact says that if $\frac{\partial f}{\partial \bar{z}_i} = 0$ for all $i = 1, \dots, n$, then for any $a = (a_1, \dots, a_n) \in W$, there exists $\epsilon_i > 0$ for $i = 1, \dots, n$ such that $f = \sum_{k_i \geq 0} c_{k_1, \dots, k_n}(z_1 - a_1)^{k_1} \dots (z_n - a_n)^{k_n}$ is a converging series for $|z_i - a_i| < \epsilon_i$. Such a function is called a holomorphic function.

Write $f(z_1, \dots, z_n) = u(z_1, \dots, z_n) + \sqrt{-1} v(z_1, \dots, z_n)$ where u and v are real-valued functions on W. If $\frac{\partial f}{\partial \bar{z}_i} = 0$, then we get the Cauchy-Riemann equations for u and v:

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i}, \quad \frac{\partial v}{\partial x_i} = -\frac{\partial u}{\partial y_i}.$$

$$\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial y_i^2} = 0, \qquad \text{and} \quad \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial y_i^2} = 0.$$

Hence u is a harmonic function on W and so is v.

Definition 1.1. Let X be a topological space with countable bases and Hausdorff. X is said to be a complex manifold if there exists a covering $\{U_i\}$ of X together with maps $\varphi_i \colon U_i \to \mathbb{C}^n$ such that φ_i is a homeomorphism from U_i to the open subset $\varphi_i(U_i)$ of $\mathbb{C}^n$ and $\varphi_i \circ \varphi_j^{-1} \colon \varphi_j(U_i \cap U_j) \subset \mathbb{C}^n \to \varphi_i(U_i \cap U_j) \subset \mathbb{C}^n$ is biholomorphic. If $\{z_1, \dots, z_n\}$ is a coordinate system for $\mathbb{C}^n$, then it is also called a local coordinate system for U_i (or X). $z_k(\varphi_i)$ is a function on U_i, by abuse of notation, denoted also by z_k.

Example 1.2. Consider the Riemann sphere $S^2 = \mathbb{P}^1$.

One way to define $\mathbb{P}^1$ is to regard it as the space of lines in $\mathbb{C}^2$ passing through the origin. Here is another way to define $\mathbb{P}^1$: $\mathbb{P}^1 = \{\mathbb{C}^2 - \{(0,0)\}\}/\mathbb{C}^*$ where C^* acts on $\mathbb{C}^2$ by $k(x,y) = (kx, ky)$ for $k \in \mathbb{C}^*$ and $(x,y) \in \mathbb{C}^2$. Hence we can write

$$\mathbb{P}^1 = \{[z_0, z_1] \mid [z_0, z_1] = [kz_0, kz_1] \text{ for } k \in \mathbb{C}^*, (z_0, z_1) \neq (0,0)\}$$

where $[z_0, z_1]$ is called the homogeneous coordinates.

Let $\pi\colon \mathbb{C}^2 - \{(0,0)\} \to \mathbb{P}^1$ be the quotient map. The topology on $\mathbb{P}^1$ is the induced quotient topology. Take two open subsets U_0 and U_1 of $\mathbb{P}^1$:

$$U_0 = \{[z_0, z_1] \in \mathbb{P}^1 \mid z_0 \neq 0\}, \qquad U_1 = \{[z_0, z_1] \in \mathbb{P}^1 \mid z_1 \neq 0\}.$$

we have homeomorphisms:

$$\varphi_0\colon U_0 \to \mathbb{C}, \quad [z_0, z_1] \to u = \frac{z_1}{z_0}; \quad \varphi_1\colon U_1 \to \mathbb{C}, \quad [z_0, z_1] \to w = \frac{z_0}{z_1}.$$

Hence over the overlap $\varphi_0(U_0 \cap U_1) = \mathbb{C}^*$, we have for $u \in \mathbb{C}^*$,

$$w = \varphi_1 \circ \varphi_0^{-1}(u) = \frac{1}{u}, \qquad u \to [1, u] = [1/u, 1] \to 1/u.$$

Clearly $\varphi_1 \circ \varphi_0^{-1}(u) = 1/u$ is holomorphic on $\mathbb{C}^*$. Therefore $\mathbb{P}^1$ is a complex manifold. Take $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\}$. $S^1 = \{e^{i\theta}\}$ acts on S^3 by the natural action. One can see that $\mathbb{P}^1 = S^3/S^1$. Since S^3 is compact, $\mathbb{P}^1$ is compact.

The following topological spaces are complex manifolds as well: $\mathbb{C}^n$, open subsets of $\mathbb{C}^n$, the projective space $\mathbb{P}^n = \{\mathbb{C}^{n+1} - (0, \ldots, 0)\}/\mathbb{C}^*$ where $k \cdot (z_0, z_1, \ldots, z_n) = (kz_0, kz_1, \ldots, kz_n)$ for $k \in \mathbb{C}^*$, $X \times Y$ if both X and Y are complex manifolds, $\mathbb{C}^n/\Gamma$ where Γ is a full rank lattice in $\mathbb{C}^n$, a subset of $\mathbb{C}^n$ defined by $Y = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid f(z_1, \ldots, z_n) = 0\}$ where f is a holomorphic function on $\mathbb{C}^n$ and rank of $(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}) = 1$. The last example can be understood using the complex version of the implicit function theorem.

Definition 1.3. Given a continuous function f on an open subset W of X. f is said to be holomorphic if for any point p on X, there exists an open neighbourhood $U \subset W$ of p and a local coordinates $\varphi\colon U \to \mathbb{C}^n$ such that $f \circ \varphi^{-1}$ is holomorphic. Let $(z_1, \ldots, z_n) \in \mathbb{C}^n$ be a local coordinate on U, we define $\frac{\partial f}{\partial z_i} = \frac{\partial f \circ \varphi^{-1}}{\partial z_i}$.

One can check that this definition is independent of the local coordinates we choose.

Proposition 1.4. *A compact connected complex manifold X has no global holomorphic functions other than constant functions.*

Proof. Let f be a holomorphic function on X. Then the real part u of f is a harmonic function on X. By the maximum principle for harmonic functions, since X is compact, u is a constant. The same is true for the imaginary part of f. Hence f is a constant. $\square$

As a corollary, any compact complex manifold of dimension bigger than zero can never be embedded holomorphically in $\mathbb{C}^N$. For if not, we can take a holomorphic function f on $\mathbb{C}^N$ not constant on X. The restriction of f to X would be a non-constant holomorphic function on X, a contradiction. We know that any smooth compact manifold X can be smoothly embedded in some $\mathbb{R}^N$ and there are lots of global smooth non-constant functions on X.

Let's come back to the projective space $\mathbb{P}^n$. $\mathbb{P}^n$ can be regarded as a compactification of $\mathbb{C}^n$ as follows. $\mathbb{P}^n$ has some open subsets

$$U_i = \{[z_0, z_1, \dots, z_n] \in \mathbb{P}^n \mid z_i \neq 0\}$$

for $i = 0, \dots, n$. All these open subsets are biholomorphic to $\mathbb{C}^n$, for example,

$$\varphi \colon U_0 \to \mathbb{C}^n, \quad \varphi([z_0, \dots, z_n]) = (\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}) \in \mathbb{C}^n.$$

The complement of U_0 is

$$\mathbb{P}^n - U_0 = \{[0, z_1 \dots, z_n] \in \mathbb{P}^n\} = \mathbb{P}^{n-1}.$$

Therefore , if we identify $\mathbb{C}^n$ with U_0 by $(x_1, \dots, x_n) \in \mathbb{C}^n \to [1, x_1, \dots, x_n]$, then the set of infinities, i.e., $\mathbb{P}^n - U_0$, is the space of lines on $\mathbb{C}^n$ passing through the origin.

Example 1.5. Consider the curve $Y \subset \mathbb{C}^2$, $Y = \{(x, y) \in \mathbb{C}^2) \mid xy = 1\}$. One can show that it is a (non-compact) complex manifold of dimension one. Take a compactitication of $\mathbb{C}^2$ as $\mathbb{C}^2 \to U_0 \subset \mathbb{P}^2$, $(x, y) \to [1, x, y]$. For the homogeneous coordinates $[z_0, z_1, z_2]$ of $\mathbb{P}^2$, we choose a homogeneous polynomial $F(z_0, z_1, z_2) = z_1 z_2 - z_0^2$. Let $\overline{Y} = \{[z_0, z_1, z_2] \in \mathbb{P}^2 \mid F(z_0, z_1, z_2) = 0\}$. Clearly $\overline{Y}$ is a closed subset of $\mathbb{P}^2$ even though F is not a well defined function on $\mathbb{P}^2$. $\overline{Y} \cap U_0 = \{[z_0, z_1, z_2] \in \mathbb{P}^2 \mid z_0 \neq 0, z_1 z_2 - z_0^2 = 0\}$ is isomorphic to Y by taking $x = z_1/z_0$, $y = z_2/z_0$.

The intersection $\overline{Y}$ with the set of infinities is

$$\overline{Y} \cap (\mathbb{P}^2 - U_0) = \{[z_0, z_1, z_1] \in \mathbb{P}^2 \mid z_0 = 0, z_1 z_2 = z_0^2\} = \{[0, 0, 1]\} \cup \{[0, 1, 0]\},$$

the two points which represent two "asymptotic" directions of Y in $\mathbb{C}^2$, i.e., x-direction and y-direction.

In general, a homogeneous polynomial F on $\mathbb{P}^n$ is not a well defined function on $\mathbb{P}^n$, but the closed subset

$$\overline{Y} = \{[z_0, z_1, \dots, z_n] \in \mathbb{P}^n \,|\, F(z_0, z_1, \dots, z_n) = 0\}$$

is well defined and is a codimension one hypersurface. The function $f(z_1, \dots, z_n) = F(1, z_1, \dots, z_n)$ is a polynomial on $\mathbb{C}^n = U_0$ and the zero locus of f, $Y = \{(z_1, \dots, z_n) \,|\, f(z_1, \dots, z_n) = 0)\}$, is $\overline{Y} \cap U_0$. Conversely, for a polynomial $f(z_1, \dots, z_n)$ of degree d on $\mathbb{C}^n$, e.g., $f(z_1, z_2) = z_1 z_2 - 1$, we can homogenize it to get a homogeneous polynomial $F(z_0, z_1, \dots, z_n)$ of degree d and $Y = \overline{Y} \cap U_0$, e.g., $F = z_1 z_2 - z_0^2$. $\overline{Y}$ is a compactification of Y.

Definition 1.6. A projective variety X is a closed subset of $\mathbb{P}^n$ defined as

$$X = \{[z_0, z_1, \dots, z_n] \in \mathbb{P}^n \,|\, F_1(z_0, \dots, z_n) = 0, \dots, F_k(z_0, \dots, z_n) = 0\}$$

where $F_1, \dots, F_k$ are homogeneous polynomials on $\mathbb{P}^n$.

An open subset U of X is called an algebraic quasi-variety. X is said to be nonsingular or smooth if it is a complex manifold.

Example 1.7. Let's look at two examples of singular curves.

Let $X = \{[z_0, z_1, z_2] \in \mathbb{P}^2 \,|\, F(z_0, z_1, z_2) = z_1^3 - z_2^2 z_0 = 0\}$. Consider $X \cap U_0 = \{(x, y) \in \mathbb{C}^2 \,|\, x^3 = y^2\}$. Let $f(x, y) = F(1, x, y)$. Then $\dfrac{\partial f}{\partial x} = 3x^2$ and $\dfrac{\partial f}{\partial y} = -2y$. Clearly X is singular at the point $[1, 0, 0]$. This singular point is called a cusp.

Let $Y = \{[z_0, z_1, z_2] \in \mathbb{P}^2 \,|\, F(z_0, z_1, z_2) = z_2^2 z_0 - z_1^2(z_1 + z_0) = 0)\}$. Consider $Y \cap U_0$. Let $f(x, y) = F(1, x, y) = y^2 - x^2(x + 1)$. $\dfrac{\partial f}{\partial x} = -3x^2 - 2x$ and $\dfrac{\partial f}{\partial y} = 2y$. So $[1, 0, 0]$ is a singular point of Y. This point is called an ordinary double point.

In the end, we list some results we won't prove.

A theorem of Chow says that any compact complex submanifold of $\mathbb{P}^n$ is algebraic, i.e., it is the zero locus of some finitely many homogeneous polynomials on $\mathbb{P}^n$.

Let X be an algebraic variety. Define a new topology, called Zariski topology, on X by defining a closed subset of X to be a subvariety of X. Note that this topology is not Hausdorff. For example, take a Riemann surface X. A closed subset of X in Zariski topology is a set of finitely many points on X.

2. Meromorphic functions, divisors and line bundles

Let X be a smooth algebraic variety, i.e., X is holomorphically embedded in some $\mathbb{P}^n$. let F and G be two homogeneous polynomials over $\mathbb{P}^n$ of the degree d. Consider the quotient

$$\frac{F}{G} = \frac{k^d F}{k^d G} = \frac{F(kz_0, \ldots, kz_n)}{G(kz_0, \ldots, kz_n)} \quad \text{for } k \in \mathbb{C}^*.$$

Hence $f = \dfrac{F}{G}$ is a well defined meromorphic function on $\mathbb{P}^n$.

Example 2.1. Consider $X = \mathbb{P}^1$. Take $F(z_0, z_1) = z_1^2$, $G(z_0, z_1) = z_0(z_0 - z_1)$. Let $f = F/G$. The zeros of f counted with multiplicity are $2p$ where $p = [1, 0]$ and the pole of f counted with multiplicity are q_1 and q_2 where $q_1 = [0, 1]$ and $q_2 = [1, 1]$. We use the symbol

$$(f) = 2p - q_1 - q_2,$$

called the divisor associated to f, to record the zeros and poles (counted with multiplicity) of f. If g is another meromorphic function on X with $(g) = 2p - q_1 - q_2$, then the meromorphic function f/g has no zeros and no poles. Hence it is must be a holomorphic function which must be a constant by the Theorem 1.4, i.e., $f = ag$ for a constant a. Therefore the divisor (f) determines the function f up to a multiple of a constant.

In general, for any meromorphic function f on a complex compact manifold X, the divisor $(f) = \sum_{i=1}^{k} m_i V_i$ is a formal sum where V_i's are codimension one subvarieties of X, f vanishes along V_i with multiplicity m_i if $m_i > 0$ and f has a pole along V_i with multiplicity m_i if $m_i < 0$. By the same argument as that of the example above, we see that the divisor (f) determines the meromorphic function f up to a constant.

Now let's give a definition of divisors in the most general context.

Definition 2.2. A divisor D on X is a formal sum $D = \sum_{i=1}^{k} m_i V_i$ where V_i's are codimension one subvarieties of X and m_i's are integers.

A divisor in general is not (f) for some meromorphic function f on X. For example, let X be $\mathbb{P}^1$, $D = p_1 + p_2$. If $D = (f)$, then f would be a holomorphic function on X since it has no poles and it is not a constant since it vanishes only at the points p_1 and p_2, a contradiction.

We define $Div(X)$ to be the set of all divisors on X. Addition, minus and the zero elmemnt can be defined on $Div(X)$ as follows:

(i) Addition: for $D = \sum m_i V_i$, $D' = \sum m'_j V'_j$, define $D + D' = \sum m_i V_i + \sum m'_j V'_j$.

(ii) Minus: $-D = \sum(-m_i)V_i$.
(iii) Zero element: $D = (1)$.

We can check that $Div(X)$ is an abelian group. Note that

$$(f \cdot g) = (f) + (g), \qquad -(f) = (\frac{1}{f})$$

for meromorphic functions f, g on X.

Here comes a question:

Question 2.3. What does a general divisor represent?

In order to answer the question, we take a different look at divisors.

Given a divisor $D = \sum m_i V_i$, for any point p on X, choose a neighbourhood U of p such that there exists a meromorphic function f on U with $U \cap D = (f)$. Hence we can get an open covering $\{U_\alpha\}$ of X together with a collection of meromorphic functions f_α over U_α. Such a collection is also called a *Cartier divisor*. Then $g_{\alpha\beta} = \dfrac{f_\alpha}{f_\beta}$ is a non-vanishing holomorphic function on $U_\alpha \cap U_\beta$. $g_{\alpha\beta}$'s satisfy the following properties:

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1} \text{ on } U_\alpha \cap U_\beta, \quad g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \tag{2.1}$$

If one knows the theory of vector bundles, one sees that the collection $\{U_\alpha, f_\alpha\}$ defines a (complex) line bundle on X.

Exercise 2.4. Any Cartier divisor defines a divisor D in the sense of the Definition 2.2.

Definition 2.5. Let X be a complex manifod. A topological space E with a continuous map π to X, called a projection, is a holomorphic (or complex) vector bundle over X if for any point $p \in X$ $E|_p$ is a complex vector space and there exists an open covering $\{U_\alpha\}$ of X such that

(i) there exists a homeomorphism $\varphi_\alpha\colon E|_{U_\alpha} = \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$ such that $\varphi_\alpha|_p\colon E|_p \to p \times \mathbb{C}^r$ is an isomorphism of complex vector spaces.
(ii) $g_{\alpha\beta}(x) = \varphi_\alpha \circ \varphi_\beta^{-1}|_{x\times\mathbb{C}^r}\colon x \times \mathbb{C}^r \to x \times \mathbb{C}^r$, called the transition function, is a holomorphic map from $U_\alpha \cap U_\beta$ to $GL(r, \mathbb{C})$.

$\varphi_\alpha\colon E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^r$ is called a trivialization of E over U_α, r is called the rank of E. One can check that $g_{\alpha\beta}$'s satisfy (2.1).

Conversely, given an open covering $\{U_\alpha\}$ of X and a collection of $g_{\alpha\beta}$'s which are holomorphic maps from $U_\alpha \cap U_\beta$ to $GL(r, \mathbb{C})$ satisfying (2.1), we can construct a holomorphic vector bundle E over X:

$$E = \coprod U_\alpha \times \mathbb{C}^r / \sim$$

where $(x, v_\alpha) \sim (y, v_\beta)$ if and only if $x = y$ and $v_\alpha = g_{\alpha\beta} v_\beta$.

Exercise 2.6. Show that such E above is a well defined holomorphic vector bundle over X.

Recall that given a divisor D, we get a Cartier divisor $\{U_\alpha, f_\alpha\}$. $g_{\alpha\beta} = f_\alpha / f_\beta$ is a nonvanishing holomorphic function on $U_\alpha \cap U_\beta$ satisfying (2.1). Hence by the discussion above, we get a (holomorphic) line bundle, denoted by $[D]$. In fact, we get something more. We can also get a meromorphic section of $[D]$ whose associated divisor is D.

Definition 2.7. Given a line bundle L. Let s be a holormorphic section of the projection $\pi\colon L \to X$ away from a codimension one subvariety such that for each $p \in X$ there exist a neighbourhood U of p and a trivialization $\varphi\colon L|_U \to U \times \mathbb{C}$ such that $\varphi(s)(x) = (x, f(x))$ where f is a meromorphic function over U.

Given a Cartier divisor $D = \{U_\alpha, f_\alpha\}$, there exists a "canonical" meromorphic section s of $[D]$ defined as follows: for the trivialization $\varphi_\alpha\colon [D]|_{U_\alpha} \to U_\alpha \times \mathbb{C}$, $s(x) = \varphi_\alpha^{-1}(x, f_\alpha(x))$ for $x \in U_\alpha$. One can check that s is a globally defined meromorphic section of $[D]$.

For a meromorphic section s of a line bundle L. we can define a divisor D associated to s, denoted by (s), as follows: take local trivializations of L over X, $\varphi_\alpha\colon L|_{U_\alpha} \to U_\alpha \times \mathbb{C}$. Let $\varphi_\alpha(s)(x) = (x, f_\alpha(x))$. f_α is a moremorphic function over U_α. One can check that $f_\alpha/f_\beta = \varphi_\alpha \circ \varphi_\beta^{-1}$. Hence f_α/f_β is a nonvanishing holomorphic function over $U_\alpha \cap U_\beta$. Therefore $\{U_\alpha, f_\alpha\}$ is a Cartier divisor.

Now we can conclude that there is a one-to-one correspondence between the set of divisors D (equivalently Cartier divisors) and the set of line bundles L with meromorphic sections s up to a constant. In another word, a divisor D corresponds to a meromorphic section s of the line bundle $[D]$.

Define $Pic(X)$ to be the set of (holomorphic) line bundles over X modulo bundle isomorphisms. $Pic(X)$ is a multiplicative abelian group:

(i) Multiplication: given L and L' in $Pic(X)$, $L \otimes L'$ is the multiplication. If $\{g_{\alpha\beta}, U_\alpha\}$ and $\{g'_{\alpha\beta}, U_\alpha\}$ are transition functions of L and L' respectively, then $\{g_{\alpha\beta} g'_{\alpha\beta}, U_\alpha\}$ are the transition functions of $L \otimes L'$.

(ii) Inverse: Given $L \in Pic(X)$, the inverse of L, denoted by L^*, is the dual bundle $Hom(L, \mathbb{C})$. The transition functions of L^* are $\{g_{\alpha\beta}^{-1}, U_\alpha\}$.

(iii) Unit element: the trivial line bundle is the unit element.

Now we get a map

$$[\quad]\colon Div(X) \to Pic(X). \tag{2.2}$$

One can check the map $[\quad]$ is a homomorphism of groups.

A deep theorem of Lefschetz on (1,1)-classes implies that the map $[\quad]$ is surjective when X is a projective manifold, i.e., X is holomorphically embedded in some projective space as a closed complex submaniifold .

The next question is what the kernel of $[\quad]$ is.

Let $D = \{U_\alpha, f_\alpha\}$ be a Cartier divisor such that $[D] = X \times \mathbb{C}$. There exist trivializations $\varphi_\alpha\colon [D]|_{U_\alpha} \to U_\alpha \times \mathbb{C}$ and the transition functions are $\varphi_\alpha \circ \varphi_\beta^{-1} = f_\alpha/f_\beta$. Take a nonzero trivial section s of $[D] = X \times \mathbb{C}$. Under the map φ_α, $\varphi_\alpha(s)(x) = (x, g_\alpha(x))$ for $x \in U_\alpha$. Similarly $\varphi_\beta(s)(x) = (x, g_\beta(x))$. Both g_α and g_β are holomorphic and nonvanishig. Therefore we have $g_\alpha(x) = \varphi_\alpha \circ \varphi_\beta^{-1}|_{x\times\mathbb{C}}(g_\beta(x)) = \dfrac{f_\alpha}{f_\beta} g_\beta$. Hence over $U_\alpha \cap U_\beta$, $\dfrac{f_\alpha}{g_\alpha} = \dfrac{f_\beta}{g_\beta}$, i.e., $\{\dfrac{f_\alpha}{g_\alpha}\}$ is a globally defined meromorphic function f on X and $D = (f)$. One can also check easily that if $D = (f)$, $[D]$ is a trivial line bundle. Therefore we get the kernel of the map $[\quad]$ is the set of global meromorphic functions on X and $Pic(X) = Div(X)/Ker[\quad]$. This gives arise to the following definition.

Definition 2.8. Given two divisors D and D' on X. D and D' are said to be linearly equivalent, denoted by $D \sim D'$, if there exists a global meromorphic function f on X such that $D = D' + (f)$. Equivalently, $D \sim D'$ if and only if $[D] = [D']$.

Now we can give a complete answer to the Question 2.3: a divisor D corresponds to a line bundle $[D]$ with a meromorphic section s and vice versa. The section s can be regarded as a "twisted" meromorphic function. A meromorphic function f corresponds to a "special" divisor linearly equivalent to 0 which corresponds to the trivial line bundle with the meromorphic section given by f.

Let's look at several examples.

Example 2.9. Universal line bundle on $\mathbb{P}^n$.

Consider a subset $L \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$:

$$\begin{aligned} L &= \{([z_0, \ldots, z_n], (\ell_0, \ldots, \ell_n)) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid \\ &\qquad (\ell_0, \ldots, \ell_n) = k(z_0, \ldots, z_n) \text{ for some } k\} \end{aligned}$$

with the projection to the first factor $\pi\colon L \to \mathbb{P}^n$. Define

$$\varphi_i\colon L|_{U_i} \to U_i \times \mathbb{C}, \quad ([z_0,\dots,z_n],(\ell_0,\dots,\ell_n)) \to ([z_0,\dots,z_n],\ell_i).$$

One can check that this is an isomorphism of vector spaces on each fiber. Over $U_i \cap U_j$ $g_{ij} = \varphi_i \circ \varphi_j^{-1}([z_0,\dots,z_n],\ell_j) = ([z_0,\dots,z_n],\frac{z_i}{z_j}\ell_j)$. Hence the transition functions are $\{\frac{z_i}{z_j}, U_i \cap U_j\}$. Therefore L is a line bundle, called the universal line bundle.

$s_0([z_0,\dots,z_n]) = ([z_0,\dots,z_n],(1,\frac{z_1}{z_0},\dots,\frac{z_n}{z_0}))$ is a meromorphic section of L whose associated divisor $(s_0) = -H$ where H is the hyperplane $H = \{z_0 = 0\}$ in $\mathbb{P}^n$. Hence $L = [-H]$. The line bundle $[H]$, which is the dual of $[-H]$, is called the hyperplane line bundle.

Example 2.10. Canonical line bundle.

Let X be a complex manifold of complex dimension equal to n, T_X^* be the holomorphic cotagent bundle of X. Define the canonical line bundle $K_X = \wedge^n T_X^*$.

On $\mathbb{P}^n$, over U_0, $\omega_0 = d(\frac{z_1}{z_0}) \wedge \ldots \wedge d(\frac{z_n}{z_0})$ is a nonvanishing holomorphic n-form, hence provides a trivialization of K_X over U_0,

$$\varphi_0\colon K_X|_{U_0} \to U_0 \times \mathbb{C}, \qquad f(p)\omega_0 \to (p, f(p))$$

where f is a holomorphic function on U_0.

Similarly, over U_1, $\omega_1 = -d(\frac{z_0}{z_1}) \wedge d(\frac{z_2}{z_1}) \wedge \ldots \wedge d(\frac{z_n}{z_1})$ provides a trivialization of K_X over U_1,

$$\varphi_1\colon K_X|_{U_1} \to U_1 \times \mathbb{C}, \qquad g(p)\omega_1 \to (p, g(p))$$

where g is a holomorphic function on U_1. Now we have

$$\begin{aligned}
\omega_1 &= -d(\frac{z_0}{z_1}) \wedge d(\frac{z_2}{z_1}) \wedge \ldots \wedge d(\frac{z_n}{z_1}) \\
&= \frac{d(z_1/z_0)}{z_1^2/z_0^2} \wedge d(\frac{z_2/z_0}{z_1/z_0}) \wedge \ldots \wedge d(\frac{z_n/z_0}{z_1/z_0}) \\
&= \frac{z_0^2}{z_1^2} d(z_1/z_0) \wedge \frac{(z_1/z_0)d(z_2/z_0) - (z_2/z_0)d(z_1/z_0)}{z_1^2/z_0^2} \wedge \ldots \\
&= (\frac{z_0}{z_1})^{n+1} d(\frac{z_1}{z_0}) \wedge \ldots \wedge d(\frac{z_n}{z_0}) \\
&= (\frac{z_0}{z_1})^{n+1} \omega_0.
\end{aligned}$$

Hence the transition function $g_{01} = (\frac{z_0}{z_1})^{n+1}$. Similarly we can obtain the other transition functions g_{ij}. Compare this with the previous Example 2.9, we see that $K_X = [-H]^{\otimes(n+1)} = [-(n+1)H]$.

Example 2.11. Adjunction formula.

Let X be a complex manifold of dimension n, $V \subset X$ be a codimension one submanifold of X. We have the following exact sequence of vector bundles

$$0 \to T_V \to T_X|_V \to N_{V/X} \to 0$$

where $N_{V/X}$ is the normal bundle of V in X. Take the dual of the exact sequence, we get

$$0 \to N^*_{V/X} \to T^*_X|_V \to T^*_V \to 0$$

where $N^*_{V/X}$ is called the conormal bundle of V in X.

Choose an open covering $\{U_\alpha\}$ of X such that, over each U_α, $V \cap U_\alpha$ is given by the zero locus $\{f_\alpha = 0\}$ of some holomorphic function f_α defined over U_α. Then $df_\alpha|_{V\cap U_\alpha}$ is a non-vanishing holomorphic section of $N^*_{V/X}$. One way to see this is to choose a local coordinates $z_1, \ldots, z_n$ in U_α such that $V \cap U_\alpha = \{z_1 = 0\}$. Hence $dz_1, \ldots, dz_n$ is a basis for $T^*_X|_{U_\alpha}$, $dz_2, \ldots, dz_n$ is a basis of $T^*_V|_{U_\alpha}$, dz_1 is a basis of $N^*_{V/X}$ and we take $f_\alpha = z_1$.

Now $df_\alpha|_{V\cap U_\alpha}$ provides a local trivialization of $N^*_{V/X}$:

$$\varphi_\alpha \colon N^*_{V/X}|_{U_\alpha} \to U_\alpha \times \mathbb{C}, \qquad g_\alpha(p) df_\alpha|_V \to (p, g_\alpha(p)).$$

When restricted to V, we get

$$df_\beta|_V = d(\frac{f_\beta}{f_\alpha} \cdot f_\alpha)|_V = d(\frac{f_\beta}{f_\alpha})|_V \cdot f_\alpha|_V + \frac{f_\beta}{f_\alpha}|_V \cdot d(f_\alpha)|_V = \frac{f_\beta}{f_\alpha}|_V \cdot d(f_\alpha)|_V.$$

Hence $g_\alpha = (\frac{f_\beta}{f_\alpha}|_V) g_\beta$, i.e., the transition function for $N^*_{V/X}$ over $U_\alpha \cap U_\beta$ is $\frac{f_\beta}{f_\alpha}|_V$ which is also the transition functions for the line bundle $[-V]|_V$. Therefore $N^*_{V/X} = [-V]|_V$. Since $\wedge^n(T^*_X|_V) = (\wedge^{n-1} T^*_V) \otimes N^*_{V/X}$, we get the so called adjunction formula

$$K_V = (K_X \otimes [V])|_V. \tag{2.3}$$

Sometimes people use K_X to denote a divisor corresponding to the canonical line bundle as well, called the canonical divisor.

Given a divisor $D = \sum m_i V_i$, we say that D is effective if and only if $m_i \geq 0$, denoted by $D \geq 0$. If D is effective, then $[D]$ has holomorphic sections. Define $H^0(X; [D])$ to be the vector space of holomorphic sections of the line bundle $[D]$. This space can be empty which means that the line bundle $[D]$ doesn't have holomorphic sections. Conversely if a line bundle L has a holomorphic section s, then the divisor $D = (s)$ is an effective divisor.

Definition 2.12. Given a divisor D, $|D|$ is defined to be the set of all effective divisors linearly equivalent to D and is called the linear system. Define $\mathcal{L}(D) = \{f \text{ meromorphic} |\, (f) + D \geq 0\}$.

Take a meromorphic section s_0 of $[D]$ such that $(s_0) = D$. Then $(fs_0) = (f) + (s_0) \geq 0$ for $f \in \mathcal{L}(D)$. Hence fs_0 is a holomorphic section of $[D]$. Therefore there is a one-to-one correspondence between $\mathcal{L}(D)$ and $H^0(X; [D])$ and $|D| = \mathbb{P}(H^0(X; [D]))$.

One of the most important usages of line bundles is to construct morphisms from line bundles. It goes as follows.

Suppose $H^0(X; [D])$ isn't empty. Let E be a subspace of $H^0(X; [D])$. Take a basis $\{s_0, \ldots, s_k\}$ of E. Let $B = \{p \in X \mid s_0(p) = 0, \ldots, s_k(p) = 0\}$. B is called the base locus of E. When $E = H^0(X, [D])$, B is also called the base locus of the linear system $|D|$, i.e., $p \in B$ if and only if $p \in D'$ for all $D' \in |D|$.

We can define a "map" $\varphi_E \colon X - \to \mathbb{P}^k$ by mapping $p \in X$ to $[s_0(p), \ldots, s_k(p)]$. To be more precise, given a point $p \notin B$, take a trivialization of L over an open subset U containing p and let $f_0, \ldots, f_k$ be the corresponding holomorphic functions of $s_0(p), \ldots, s_k(p)$ respectively under the trivialization and define $\varphi_E(p) = [f_0(p), \ldots, f_k(p)]$. One can check that the definition is independent of trivializations. Clearly this map is only defined over $X - B$ and is called a rational map in general. It is a holomorphic map on $X - B$ and the image of φ_E doesn't lie on any hyperplane in $\mathbb{P}^k$, called non-degenerate.

Definition 2.13. $\varphi \colon X - \to Y$ is called a rational map between two varieties X and Y if there exists a subvariety V of X such that $\varphi \colon X - V \longrightarrow Y$ is a holomorphic map.

Suppose $B = \emptyset$, then we get a morphism $\varphi_E \colon X \to \mathbb{P}^k$. Choosing a different basis of E amounts to a projective automorphism of $\mathbb{P}^k$. One can check that $\varphi^*[H] = [D]$ where $[H]$ is the hyperplane line bundle on $\mathbb{P}^k$.

Conversely, if we have a non-degenerate map $f \colon X \to \mathbb{P}^k$. Take $z_0, \ldots, z_k$ as the basis of $H^0(\mathbb{P}^k; [H])$. Then $f^*z_0, \ldots, f^*z_k$ form a basis of a subspace E of $H^0(X; f^*[H])$. Hence we get a one-to-one correspondence between the set of non-degenerate maps $f \colon X \to \mathbb{P}^k$ modulo projective transformations and the set of line bundles L over X with a $k+1$-dimensional subspace E of $H^0(X;L)$ such that E has no base locus.

Question 2.14.

(i) Given a line bundle, how do we compute the dimension of $H^0(X;L)$?
(ii) When is the linear system $|D|$ base point free?
(iii) When is the map φ_E an embedding?

In order to answer the questions above, we need Riemann-Roch Theorem, Serre duality and Kodaira vanishing Theorem all of which depend on the cohomology theory of sheaves.

Finally, let's list some results which we won't prove.

Theorem 2.15 (Bertini's Theorem)**.** *Let X be a compact complex submanifold of $\mathbb{P}^n$. There exists a hyperplane $H \subset \mathbb{P}^n$ such that $V = X \cap H$ is a complex submanifold of X.*

Theorem 2.16 (Lefschetz Hyperplane Theorem)**.** *With the same assumption as in Theorem 2.15. Then the map $H^q(X,\mathbb{Q}) \to H^q(V,\mathbb{Q})$ induced by the inclusion $V \to X$ is an isomorphism for $q \le n-2$ where n is the complex dimension of X.*

Using Bertini's theorem, we can construct many projective manifolds. Let $\varphi_d \colon \mathbb{P}^n \to \mathbb{P}^N$ be the d-uple embedding,

$$\mathbb{P}^d \to \mathbb{P}^N, \quad [z_0,\dots,z_n] \to [u_0,\dots,u_N]$$

where $\{u_0,\dots,u_N\}$ is the collection of all monomials such as $z_0^d, z_0^{d-1}z_1,\dots,z_n^d$. By abuse of notations, we also use $[u_0,\dots,u_N]$ as the homogeneous coordinates of $\mathbb{P}^N$. By Bertini's theorem, take a hyperplane $H = \{a_0u_0 + \dots a_Nu_N = 0\}$ of $\mathbb{P}^N$ such that $H \cap \varphi_d(\mathbb{P}^n)$ is a submanifold of $\varphi_d(\mathbb{P}^n)$. $H \cap \varphi_d(\mathbb{P}^n)$ is isomorphic to a smooth hypersurface Y of $\mathbb{P}^n$ given by a degree d homogeneous polynomial $F = a_0z_0^d + \dots + a_Nz_n^d$. We can use the adjunction formula to calculate the canonical line bundle K_Y.

First of all, the line bundle $[Y] \cong [dH_0]$ where H_0 is the hyperplane $\{z_0 = 0\}$. This is because the meromorphic function $\dfrac{F}{z_0^d}$ has its associated divisor to be $Y - dH_0$. Thus divisors Y and dH_0 are linearly equivalent.

By the adjunction formula,

$$K_Y \cong \big(K_{\mathbb{P}^n} \otimes [Y]\big)|_Y \cong \big([-(n+1)H_0] \otimes [dH_0]\big)|_Y = [(d-n-1)H_0]|_Y.$$

Let $n = 2$. When $d = 1$, Y is a line in $\mathbb{P}^2$. When $d = 2$, Y is a conic curve still isomorphic to $\mathbb{P}^1$. This can be seen via the 2-uple embedding: $[x,y] \in \mathbb{P}^1 \to [x^2, xy, y^2] \in \mathbb{P}^2$. If we use $[u,w,v]$ as the homogeneous coordinates of $\mathbb{P}^2$, the image of the 2-uple embedding is given by $uv - w^2 = 0$, a conic curve. Other conic curves can be mapped to this conic via automorphisms in $PGL(2) = \mathbb{P}\big(GL(3,\mathbb{C})\big)$. Therefore in both cases above, K_Y has no global non-zero sections. When $d = 3$, $K_Y \cong Y \times \mathbb{C}$ and thus $dim\Gamma(Y, K_Y) = 1$. This is the cubic curve which is an elliptic curve. When $d \geq 4$, $K_Y \cong [(d-3)H_0]|_Y$ and $dim\Gamma(Y, K_Y) \geq 1$. In fact, one can calculate that $dim\Gamma(Y, K_Y) = (d-1)(d-2)/2 > 1$.

Let $n = 3$. When $d = 1, 2, 3$, $K_Y \cong [(d-4)H_0]|_Y$ has no non-zero global sections. For $d = 1$, Y is just isomorphic to $\mathbb{P}^2$. For $d = 2$, the quadric surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. This can be seen as follows. Consider the map $f\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, $[x,y] \times [u,v] \to [xu, xv, yu, yv]$. If we use $[z_0, z_1, z_2, z_3]$ as the homogenous coordinates of $\mathbb{P}^3$, the image of the map f is given by the equation $z_0 z_3 - z_1 z_2 = 0$, i.e., a quadric surface. When $d = 4$, $K_Y \cong Y \times \mathbb{C}$ is a trivial line bundle, similar to the elliptic curve for dimensional one case. Such a surface is called K3 surface. When $d > 4$, $K_Y \cong [(d-4)H_0]|_Y$ with $d - 4 > 0$. Such surfaces are called general type.

Let $n = 4$, When $d = 1, 2, 3, 4$, $K_Y \cong [(d-5)H_0]|_Y$ has no non-zero global sections. When $d = 5$, $K_Y \cong Y \times \mathbb{C}$ is a trivial line bundle, similar to K3 surfaces. It is called the Calabi-Yau three-fold. When $d > 5$, Y is called general type and $K_Y \cong [(d-5)H_0]|_Y$ with $d - 5 > 0$.

3. Sheaves and cohomologies of sheaves

Example 3.1. Let X be a complex manifold, U be an open subset of X. Let $\mathcal{O}(U)$ be the set of holomorphic functions on U. $\mathcal{O}(U)$ is an abelian group. For two open subsets $U \subset V$, the restriction map

$$r_{V,U}\colon \mathcal{O}(V) \to \mathcal{O}(U), \quad r_{V,U}(f) = f|_U$$

is a group homomorphism. $r_{U,U}$ is an identity map. We have the following properties:

(i) For any triple of open subsets $U \subset V \subset W$, we have $r_{W,U} = r_{V,U} \circ r_{W,V}$.

(ii) For a collection of open sets $U_\alpha \subset X$, let $U = \cup_\alpha U_\alpha$. If $h \in \mathcal{O}(U)$ and $r_{U,U_\alpha}(h) = 0$, then $h = 0$.

(iii) If $f_\alpha \in \mathcal{O}(U_\alpha)$ and if $r_{U_\alpha, U_\alpha \cap U_\beta}(f_\alpha) = r_{U_\beta, U_\alpha \cap U_\beta}(f_\beta)$, then there exists $h \in \mathcal{O}(U)$ such that $r_{U,U_\alpha}(h) = f_\alpha$.

We define the stalk $\mathcal{O}_{X,p}$ to be the group

$$\{(f, U) \mid f \in \mathcal{O}(U), U \text{ is an open subset containing } p\}/\sim$$

where the equivalence relation $\sim$ is defined as $(f, U) \sim (g, V)$ if and only if there exists an open subset W containing p, $W \subset U \cap V$ such that $f|_W = g|_W$. The stalk $\mathcal{O}_{X,p} = \{$ converging power series at p $\}$.

So we have seen an example of a sheaf.

Definition 3.2. $\mathcal{F}$ is called a sheaf over X if for any open subset U of X, there exists an abelian group $\mathcal{F}(U)$. For any two subsets $U \subset V$. there exists a restriction map $r_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ which is a group homomorphism satisfying the properties (i), (ii) and (iii) above with $\mathcal{O}$ replaced by $\mathcal{F}$ and an additional property $\mathcal{F}(\emptyset) = 0$ and $r_{UU} = id$.

Any element $f \in \mathcal{F}(U)$ is called a section of $\mathcal{F}$ over U.

The sheaf $\mathcal{O}_X$ we constructed in the Example 3.1 is called the structure sheaf of X. Note that $\mathcal{O}_X$ is also a sheaf of rings since each $\mathcal{O}_X(U)$ is a ring.

If, in addition, $F(U)$ is an $\mathcal{O}_X(U)$-module for any open subset U, and the restriction maps $r_{V,U}$ are compatible with module structures, then $\mathcal{F}$ is called a sheaf of $\mathcal{O}_X$-module.

We can define the stalk of the sheaf $\mathcal{F}$ at a point p as

$$\mathcal{F}_p = \{(f, U) \mid f \in \mathcal{F}(U), U \text{ is an open subset containing } p\}/\sim$$

where the equivalence relation $\sim$ is defined as $(f, U) \sim (g, V)$ if and only if there exists an open subset W containing p, $W \subset U \cap V$ such that $r_{U,W}(f) = r_{V,W}(g)$.

Example 3.3.

(i) The constant sheaf $\mathbb{Z}$ is defined as $\mathbb{Z}(U) = \mathbb{Z}$ for any connected open subset U and the restriction map is the natural one.

(ii) Ω^p_X: $\Omega^p_X(U) = \{$holomorphic p-forms on $U\}$ and the restriction map is the natural one.

(iii) The ideal sheaf $\mathcal{I}_S$ of a subvariety S of X:

$$\mathcal{I}_S(U) = \{\text{holomorphic functions on } U \text{ vanishing on } S \cap U\}$$

and the restriction map is the natural one.

(iv) $\mathcal{O}^*_X$: $\mathcal{O}^*_X(U)$ is the multiplicative group of nonvanishing holomorphic functions on U.

Let $\pi \colon E \to X$ be a holomorphic vector bundle over X. There is a sheaf associated with E, denoted by $\mathcal{O}_X(E)$, defined as

$$\mathcal{O}_X(E)(U) = \{\text{holomorphic sections of } E|_U\}.$$

One can check that $\mathcal{O}_X(E)$ is a sheaf of $\mathcal{O}_X$-module. Moreover, for any point $p \in U$, take a local trivialization $\varphi \colon E|_U \to U \times \mathbb{C}^r$ where r is the rank of E. Hence a holomorphic section σ of $E|_U$ can be written as $(f_1, \dots, f_r)$ where f_i is holomorphic over U. Therefore $\mathcal{O}_X(E)(U)$ is isomorphic to $\mathcal{O}_X(U) \oplus \dots \oplus \mathcal{O}_X(U)$ as modules. We call this type of sheaves *locally free*. Hence a holomorphic vector bundle corresponds to a locally free sheaf. Converse is also true, i.e., a locally free sheaf $\mathcal{E}$ corresponds to a holomorphic vector bundle E such that $\mathcal{E} = \mathcal{O}_X(E)$. So sometimes we don't distinguish the difference between E and $\mathcal{O}_X(E)$.

Example 3.4. The trivial line bundle $X \times \mathbb{C}$ corresponds to the structure sheaf $\mathcal{O}_X$

Given a divisor D, it corresponds to the line bundle $[D]$ which corresponds to the rank-1 locally free sheaf $\mathcal{O}_X([D])$. By abuse of the notation, we shall use $\mathcal{O}_X(D)$ to denote $\mathcal{O}_X([D])$ and call it an invertible sheaf.

Definition 3.5. Given two sheaves $\mathcal{F}$ and $\mathcal{G}$ on X. $\varphi\colon \mathcal{F} \to \mathcal{G}$ is called a sheaf morphism if for any open subset U of X, there exists a homomorphism of groups $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ which is compatible with the restriction maps, i.e., $r_{V,U} \circ \varphi_V = \varphi_U \circ r_{V,U}$.

If, in addition, $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules and φ_U is a morphism of $\mathcal{O}_X(U)$-modules, then φ is called a morphism of sheaves of $\mathcal{O}_X$-modules.

Example 3.6. Let S be a subvariety of X. The inclusion map $\mathcal{I}_S \to \mathcal{O}_X$ is a morphism of sheaves of $\mathcal{O}_X$-modules.

The inclusion map $\varphi_U\colon \mathbb{Z}(U) \to \mathcal{O}_X(U)$ provides a morphism of sheaves from the constant sheaf $\mathbb{Z}$ to the structure sheaf.

The exponential map

$$exp_U\colon \mathcal{O}_X(U) \to \mathcal{O}_X^*(U), \quad f \in \mathcal{O}_X(U) \to e^{2\pi i f} \in \mathcal{O}_X^*(U)$$

is a morphism of sheaves from $\mathcal{O}_X$ to $\mathcal{O}_X^*$.

Given a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ between two sheaves, it is easy to see that it induces a morphism $\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ between the stalks of the sheaves at a point $p \in X$. φ is called *injective* (or *surjective*) if φ_p is injective (surjective respectively) for every point $p \in X$. One can show as an exercise that φ is injective if and only if for any open subset U of X, the map $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. The similar statement for surjection doesn't hold.

Example 3.7. The inclusion map $\mathbb{Z} \to \mathcal{O}_X$ is an injection. The exponential map $exp\colon \mathcal{O}_X \to \mathcal{O}_X^*$ is surjective. This can be checked as follows. Let (f, U) be an element in $\mathcal{O}_{X,p}^*$. We can assume that U is simply connected. Hence there exists $g \in \mathcal{O}_X(U)$ such that $f = e^{2\pi i g}$. Hence $exp_p((g, U)) = (f, U)$. In fact, the following sequence is exact, i.e., (use as a definition) it is an exact sequence for stalks at every point of X,

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0. \tag{3.1}$$

The exactness at the middle term is easy to check.

If we take $X = \mathbb{C}$, take $U = \mathbb{C} - 0$. Then $z \in \mathcal{O}_X^*(U)$. But there doesn't exist any $f \in \mathcal{O}_X(U)$ such that $exp(f) = z$. Thus surjection of a sheaf morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ doesn't imply $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is surjection.

Example 3.8. Let S be a subvariety (not necessarily smooth) of X, but we assume that it is smooth for the simplicity. One can check the following exact sequence is exact:

$$0 \to \mathcal{I}_S \to \mathcal{O}_X \to \mathcal{O}_S \to 0$$

where the last morphism is the restriction map from X to S.

Assume S is of codimension one (the following statement is also true when S is singular). Let s_0 be the holomorphic section of $[S]$. Since $\mathcal{O}_X(-S) = \mathcal{O}_X(S)^*$, we will have a morphism $s_0\colon \mathcal{O}_X(-S) \to \mathcal{O}_X$. One can check that the image of this map is $\mathcal{I}_S$. Hence we get another exact sequence

$$0 \to \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0 \tag{3.2}$$

Now let's define the Cech cohomology of sheaves.

Let $\mathcal{F}$ be a sheaf on X, $\underline{U} = \{U_\alpha\}$ be a locally finite open covering of X. We define the set of p-cochains as follows: $C^0(\underline{U}, \mathcal{F}) = \prod_\alpha \mathcal{F}(U_\alpha)$, $C^1(\underline{U}, \mathcal{F}) = \prod_{\alpha\neq\beta} \mathcal{F}(U_\alpha \cap U_\beta)$, etc. If $\sigma = \{\sigma_{\alpha_0 \ldots \alpha_p}\} \in C^p(\underline{U}, \mathcal{F})$, we require that $\sigma_{\alpha_0\ldots\alpha_p} = (-1)^{sign(\tau)}\sigma_{\tau(\alpha_0)\ldots\tau(\alpha_p)}$ where τ is a permutation on $p+1$ letters. There is an operator

$$\delta\colon C^p(\underline{U}, \mathcal{F}) \to C^{p+1}(\underline{U}, \mathcal{F})$$

defined as follows:

$$(\delta\sigma)_{\alpha_0\ldots\alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{\alpha_0\ldots\hat{\alpha}_j\ldots\alpha_{p+1}}|_{U_{\alpha_0}\cap\ldots\cap U_{\alpha_{p+1}}}$$

where $\hat{\alpha}_j$ means that this term is deleted.

A p-cochain σ is called a cocycle if $\delta\sigma = 0$, and a coboundary if $\sigma = \delta\tau$ for some $(p-1)$-cochain τ. We can check that $\delta^2 = 0$. Hence $\mathrm{Im}(\delta\colon C^{p-1} \to C^p)$ is contained in $Z^p(\underline{U}, \mathcal{F}) = Ker(\delta\colon C^p \to C^{p+1})$. We define

$$H^p(\underline{U}, \mathcal{F}) = \frac{Z^p(\underline{U}, \mathcal{F})}{\delta(C^{p-1}(\underline{U}, \mathcal{F}))}.$$

If $\underline{U}$ is a "good" covering, then $H^p(\underline{U}, \mathcal{F})$ is independent of the covering and called the cohomology of the sheaf $\mathcal{F}$, denoted by $H^p(X; \mathcal{F})$. We use $h^k(X; \mathcal{F})$ to denote the dimension of the vector space $H^k(X; \mathcal{F})$ if $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-module.

Let $\{\sigma_\alpha\} \in Ker\delta = H^0(X; \mathcal{F})$. Then

$$\delta(\sigma)_{\alpha\beta} = \sigma_\beta|_{U_\alpha \cap U_\beta} - \sigma_\alpha|_{U_\alpha \cap U_\beta}.$$

Thus by the definition of sheaves, there exists a section $s \in \mathcal{F}(X)$ such that $s|_{U_\alpha} = \sigma_\alpha$. Therefore $H^0(X; \mathcal{F}) = \mathcal{F}(X) = \Gamma(X; \mathcal{F})$.

One sees that $H^0(X; \mathcal{O}_X(D))$ is the space of holomorphic sections of the line bundle $[D]$, or $O_X(D)(X)$.

Using the definition, we can check that $Pic(X) = H^1(X; \mathcal{O}_X^*)$ as follows. Take a line bundle L, choose an open cover U_α of X such that $\varphi_U : L_{U_\alpha} \to U_\alpha \times \mathbb{C}$ is a trivialization. Thus we get transition functions $\{g_{\alpha\beta}\}$ satisfying

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1, \quad g_{\alpha\beta} = g_{\beta\alpha}^{-1}.$$

The collection $\{U_\alpha, g_{\alpha\beta}\}$ gives an element g in $C^1(\underline{U}, \mathcal{O}_X^*)$ such that

$$(\delta g)_{\alpha\beta\gamma} = g_{\beta\gamma} \cdot g_{\alpha\gamma}^{-1} \cdot g_{\beta\gamma} = g_{\beta\gamma} \cdot g_{\gamma\alpha} \cdot g_{\alpha\beta} = 1.$$

Thus g is a cocycle and hence gives a cohomology class in $H^1(X; \mathcal{O}_X^*)$ and hence a map from $Pic(X)$ to $H^1(X; \mathcal{O}_X^*)$.This definition is well-defined due to the fact that different choice of trivilization gives another cocycle different from the previous one by a coboundary. Then one can prove that this map is a bijection.

One of the basic properties of the cohomology of sheaves is the following result.

Theorem 3.9. *Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be a short exact sequence of sheaves on X. Then there exists a long exact sequence of cohomologies:*

$$\to H^i(X; \mathcal{E}) \to H^i(X; \mathcal{F}) \to H^i(X; \mathcal{G}) \to H^{i+1}(X; \mathcal{E}) \to \tag{3.3}$$

where $i \geq 0$.

Let's review some results from Hodge theory.

Let X be a projective nonsingular variety. Then the Hodge Decomposition Theorem says that $H^k(X; \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(X)$, $H^{p,q}(X) = \overline{H^{q,p}(X)}$, and $H^{p,q}(X) = H^q(X; \Omega_X^p)$. We use $h^{p,q}$ to denote the dimension of the vector space $H^{p,q}(X)$.

Corollary 3.10. *The Betti numbers $b_{2k+1}(X)$ of odd degree are even.*

Corollary 3.11. *$H^q(\mathbb{P}^n; \Omega^p)$ is zero if $p \neq q$ and is $\mathbb{C}$ otherwise.*

Proof. Since $H^{2k+1}(\mathbb{P}^n;\mathbb{Z})=0$, $H^q(\mathbb{P}^n,\Omega^p)=0$ if $p+q$ is odd. Since $H^{2k}(\mathbb{P}^n;\mathbb{Z})=\mathbb{Z}$, $1=b_{2k}=\sum_{p+q=2k} h^{p,q}\geq h^{p,2k-p}+h^{2k-p,p}=2h^{p,2k-p}$ if $p\neq k$. Hence $h^{2k-p.p}=0$ if $p\neq k$ and $h^{k,k}=1$. □

Finally let's list some terminologies and some facts.

Facts and Terminologies 3.12. Let X be a nonsingular projecrive variety of dimension n.

(i) $h^{n,0}=\dim H^0(X;K_X)=\dim H^n(X;\mathcal{O}_X)$ is called the geometric genus of X, denoted by p_g.

(ii) $h^{1,0}=h^0(X;\Omega_X)=h^1(X;\mathcal{O}_X)$ is called the irregularity of X, denoted by q. From Hodge theory, $2q=b_1$.

(iii) For a sheaf $\mathcal{F}$, the Euler characteristic of $\mathcal{F}$ is defined as

$$\chi(\mathcal{F})=h^0(X;\mathcal{F})-h^1(X;\mathcal{F})+\ldots+(-1)^nh^n(X;\mathcal{F}).$$

(iv) $P_m=h^0(X;\mathcal{O}_X(K_X^{\otimes m}))$ is called the plurigenera.

(v) Let $\mathcal{F}$ be a locally free sheaf on X, then $H^i(X;\mathcal{F})=H^{n-i}(X;\mathcal{F}^*\otimes\mathcal{O}_X(K_X))^*$. This is a speicial case of Serre duality.

(vi) Let $f\colon X\to\mathbb{P}^n$ be a holomorphic embedding. Let $[H]$ be the hyperplane line bundle on $\mathcal{P}^n$. We will use $\mathcal{O}_{\mathbb{P}^n}(1)$ to denote the invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(H)$. Let $L=f^*[H]$. Such a line bundle is called very ample. Any line bundle $\tilde{L}$ on X such that $\tilde{L}^{\otimes m}=L$ for some $m>0$ is called ample. If L is ample and D is a divisor, then there exists n_0 such that $D\otimes L^n$ is very ample for $n\geq n_0$. If L is ample, then for any line bundle E, there exists an integer n_0 such that $h^i(X;E\otimes L^{\otimes n})=0$ for $n\geq n_0$ and $i>0$ and $E\otimes L^{\otimes n}$ is very ample. Kodaira Vanishing Theorem says that if $\tilde{L}$ is ample, then $H^q(X;\Omega^p_X\otimes\mathcal{O}_X(\tilde{L}))=0$ if $p+q>n$.

4. Riemann surface

Let $D=\sum m_ip_i$ be a divisor on a Riemann surface S. We define the *degree* of the divisor D to be $\deg D=\sum m_i$. If f is a meromorphic function on S, then f can be regarded as a holomorphic map from S to $\mathbb{P}^1$. The associated divisor (f) is just $f^{-1}(0)-f^{-1}(\infty)$ with multiplicity considered. Hence the degree of the divisor (f) is the number of zeros of f minus the number of poles of f counted with multiplicity. Each of these numbers equals the degree of the map f. Hence the degree of (f) is zero. This implies that the degree is invariant under linearly equivalence. Therefore we can define the degree of a line bundle L, denoted by $\deg L$, to be the degree of a divisor D such that $[D]\cong L$.

Let $f\colon S\to S'$ be a non-constant holomorphic map between two Riemann surfaces S and S'. For any point $q\in S$, there exists a local coordinate z at q and w at $p = f(q) \in S'$ such that f is locally of the form $w = z^{\mu}$. $\mu(q) = \mu$ is called the *ramification index* of q. If $\mu(q) > 1$, $p = f(q)$ is called the branch point of f. $R = \sum_{q \in S} (\mu(q) - 1) q$ is called the ramification divisor. Away from the ramification divisor, f is an unramified covering (topological covering).

Let $D = \sum m_i q_i$ be a divisor on S. $f\colon S \to S'$ be a non-constant holomorphic map. Let $p \in S'$ be a point, $f^{-1}(p) = \{q_1, \ldots, q_s\}$. Define $f^*(q) = \sum_{j=1}^{s} \mu(q_j) q_j$. We extend the definition f^* to divisors by linearity. Now we get a map

$$f^*\colon Div(S') \to Div(S), \quad D \to f^*D.$$

It is easy to show that if $D \sim D'$, $f^*D \sim f^*D'$ and $[f^*D] = f^*[D]$.

Now we want to relate the genus of S to that of S'.

Take a meromorphic section σ of $K_{S'}$. $f^*\sigma$ is a meromorphic section of K_S. Around $p = f(q) \in S'$, f is locally of the form $w = z^{\mu}$. Write σ locally around p as $\dfrac{h(w)}{g(w)} dw$ where $h(w)$ and $g(w)$ are holomorphic functions around p. Write $\left(\dfrac{h(z)}{g(z)}\right)_p = \ell p$ as divisors. Around p, $f^*\sigma$ is of the form

$$\frac{h(z^{\mu})}{g(z^{\mu})} dz^{\mu} = \mu z^{\mu - 1} \frac{h(z^{\mu})}{g(z^{\mu})} dz.$$

$$(f^*\sigma)_q = (\mu - 1) q + \mu \ell q = (\mu - 1) q + f^*(\ell q) = (\mu - 1) q + f^*(\sigma)_q.$$

Therefore, summing over all $q \in S'$, we get

$$(f^*\sigma) = R + f^*(\sigma).$$

In terms of line bundles, we get

$$K_S \cong [R] \otimes f^* K_{S'}. \tag{4.1}$$

If we compute the degree, using that $\deg K_S = 2g(S) - 2$ and $\deg K_{S'} = 2g(S') - 2$ (we won't prove it here), we get

$$2g(S) - 2 = \deg R + \deg f^* K_{S'} = \deg R + n(2g(S') - 2) \tag{4.2}$$

where n is the degree of the map f. So

$$\chi(S) = n\chi(S') - \sum_{q \in S} (\mu(q) - 1). \tag{4.3}$$

These formulae are called the Riemann-Hurwitz formulae.

Here are some applications of the Riemann-Hurwitz formulae.

Corollary 4.1. *If $f\colon S \to S'$ is not a constant map, then $g(S) \geq g(S')$.*

Theorem 4.2 (Riemann-Roch)**.** *Let S be a Riemann surface of genus g, D be a divisor on S. Then we have*

$$h^0(S;\mathcal{O}_S(D)) - h^1(S;\mathcal{O}_S(D)) = \deg D + 1 - g.$$

By Serre duality, the Riemann-Roch formula can also be written as

$$h^0(S;\mathcal{O}_S(D)) - h^0(S;\mathcal{O}_S(K_S - D)) = \deg D + 1 - g. \qquad (4.4)$$

Lemma 4.3. *Let D be a divisor on S, if $\deg D < 0$, then $h^0(S;\mathcal{O}_S(D)) = 0$.*

Let's look at some applications of Riemann-Roch formula.

Example 4.4. Let S be a Riemann surface with genus $g = 0$. Let p be a point on S. By the Riemann-Roch formula, we have

$$h^0(S;\mathcal{O}_S(p)) - h^0(S;\mathcal{O}_S(-p + K_S)) = 1 + 1 = 2.$$

Since $\deg(K_S - p) = -3 < 0$, $h^0(S;\mathcal{O}_S(-p + K_S)) = 0$. Hence $h^0(S;\mathcal{O}_S(p)) = 2$. Take two linearly independent sections $s_0, s_1 \in H^0(S;\mathcal{O}_S(p))$. $H^0(S;\mathcal{O}_S(p))$ is clearly base-point-free. Then s_0/s_1 defines a holomorphic map to $\mathbb{P}^1$ with degree equal to one. Hence it is an isomorphism. Therefore every genus zero Riemann surface is isomorphic to $\mathbb{P}^1$.

Exercise 4.5. Let S be a Riemann surface, $p_1, \ldots, p_r \in S$ be points on S. Then there is a meromorphic function on S having poles (of some order ≥ 1) at each of the p_i and holomorphic elsewhere.

Exercise 4.6. Let S be a genus two Riemann surface. The canonical linear system $\mathbb{P}(H^0(S;\mathcal{O}_S(K_S)))$ determines a morphism $f\colon S \to \mathbb{P}^1$ of degree 2. Show that it is ramified at exactly six points with ramification index 2 at each point. f is uniquely determined up to an automorphism of $\mathbb{P}^1$. So S determines an (unordered) set of 6 points of $\mathbb{P}^1$ up to automorphisms of $\mathbb{P}^1$.

Theorem 4.7. *Let D be a divisor on a Riemann surface S of genus g.*

(i) *If $\deg D \geq 2g$, then the linear system $|D|$ has no base points.*
(ii) *If $\deg D \geq 2g + 1$, then $\varphi_{|D|}$ is an embedding.*

Proof. For (i), let p be a point on S, consider the exact sequence

$$0 \to \mathcal{O}_S(-p) \to \mathcal{O}_S \to \mathcal{O}_p \to 0.$$

Tensor the exact sequence by $\mathcal{O}_X(D)$, we get

$$0 \to \mathcal{O}_S(D - p) \to \mathcal{O}_S(D) \to \mathcal{O}_S(D)|_p \to 0. \qquad (4.5)$$

Take the cohomologies of the sheaves in the exact sequence (5.5), we get an exact sequence

$$H^0(\mathcal{O}_S(D)) \to H^0(\mathcal{O}_S(D)|_p) \to H^1(\mathcal{O}_S(D-p)).$$

Since $\deg(p - D + K_S) \le 1 - 2g + 2g - 2 = -1$, $h^1(\mathcal{O}_S(D-p)) = h^0(\mathcal{O}_S(K_S - D + p)) = 0$. Hence the map $H^0(\mathcal{O}_S(D)) \to H^0(\mathcal{O}_S(D)|_p)$ is a surjection. Therefore there exists a section $s \in H^0(\mathcal{O}_S(D))$ such that $s(p) \neq 0$. Thus the linear system $|D|$ is base point free.

For (ii), take two distinct points p, q on S. Consider the exact sequence

$$0 \to \mathcal{O}_S(-p-q) \to \mathcal{O}_S \to \mathcal{O}_p \oplus \mathcal{O}_q \to 0.$$

Tensor it by $\mathcal{O}_S(D)$, we get another exact sequence

$$0 \to \mathcal{O}_S(D-p-q) \to \mathcal{O}_S(D) \to \mathcal{O}_S(D)|_p \oplus \mathcal{O}_S(D)|_q \to 0. \quad (4.6)$$

Take the cohomologies of the sheaves in the exact sequence (4.6), we get an exact sequence

$$H^0(\mathcal{O}_S(D)) \to H^0(\mathcal{O}_S(D)|_p) \oplus H^0(\mathcal{O}_S(D)|_q) \to H^1(\mathcal{O}_S(D-p-q)).$$

Since $\deg(p + q - D + K_S) \le 2 - 2g - 1 + 2g - 2 = -1$, $h^1(\mathcal{O}_S(D - p - q)) = h^0(\mathcal{O}_S(K_S - D + q + p)) = 0$. Hence the map $H^0(\mathcal{O}_S(D)) \to H^0(\mathcal{O}_S(D)|_p) \oplus H^0(\mathcal{O}_S(D)|_q)$ is a surjection. Therefore there exists a section $s \in H^0(\mathcal{O}_S(D))$ such that $s(p) = 0$ but $s(q) \neq 0$. Thus $\varphi_{|D|}(p) \neq \varphi_{|D|}(q)$, i.e., the map $\varphi_{|D|}$ is one-to-one.

Let $p \in S$, consider the exact sequence

$$0 \to \mathcal{O}_S(-2p) \to \mathcal{O}_S \to \mathcal{O}_{2p} \to 0.$$

Tensor it with $\mathcal{O}_S(D)$ and take cohomologies, we get

$$H^0(\mathcal{O}_S(D)) \to H^0(\mathcal{O}_S(D)|_{2p}) \to H^1(\mathcal{O}_S(D-2p)).$$

By the same argument as above, the map $H^0(\mathcal{O}_S(D)) \to H^0(\mathcal{O}_S(D)|_{2p})$ is surjective. Hence there exists a section $s_0 \in H^0(\mathcal{O}_S(D))$ such that $s_0(p) = 0$ and p is a simple zero for s_0. Let $s_1, \ldots, s_n$ be sections of $H^0(\mathcal{O}_S(D))$ such that $\{s_0, s_1, \ldots, s_n\}$ is a basis of $H^0(\mathcal{O}_S(D))$. Without loss of generality, assume $s_n(p) \neq 0$. So

$$\varphi_{|D|}(x) = [s_0(x), s_1(x), \ldots, s_n(x)] \in \mathbb{P}^n, \quad \text{and} \quad \varphi_{|D|}(p) \in U_n.$$

So near p, we have

$$\varphi_{|D|} \colon \varphi_{|D|}^{-1}(U_n) \to U_n, \quad x \to (\frac{s_0(x)}{s_n(x)}, \ldots, \frac{s_{n-1}(x)}{s_n(x)}).$$

Since $s_0(x)$ vanishes at p only once, if z is a coordinate near p, then $s_0(x)/s_n(x) = h(z)z$ where $h(p) \neq 0$. Therefore the differential of the map $\varphi_{|D|}$ has rank equal to one at p. So it is an embedding. Note that another choice of the basis of $H^0(\mathcal{O}_S(D))$ gives another map to $\mathbb{P}^n$ which is related to the original one by an automorphism of $\mathbb{P}^n$. □

Example 4.8. Let S be an elliptic curve. Let D be a divisor on S of degree three. The canonical divisor K_S is trivial. Hence from Riemann-Roch, we get

$$h^0(S;\mathcal{O}_S(D)) - h^0(S;\mathcal{O}_S(-D)) = \deg D + 1 - g = 3.$$

Since $\deg(-D) = -3 < 0$, $h^0(S;\mathcal{O}_S(-D)) = 0$. Hence the map $\varphi_{|D|}$ embeds S into $\mathbb{P}^2$. So every elliptic curve is a plane curve.