BOUNDED ORBITS OF HOMOGENEOUS DYNAMICS

JINPENG AN AND LIFAN GUAN

Abstract. Homogeneous dynamics is a special kind of dynamics given by Lie groups, and is closely related to Diophantine approximation and other areas in number theory. In this paper, we survey main problems and results concerning bounded orbits of homogeneous dynamics, and discuss their relations with Diophantine approximation problems including the Oppenheim conjecture, the Littlewood conjecture, and the Schmidt conjecture.

1. Introduction

Homogeneous dynamics is a special kind of dynamics. Its phase space is a homogeneous space of a Lie group, and the dynamical system is induced by the group multiplication. More precisely, let G be a Lie group[1], $\Gamma \subset G$ be a closed subgroup, and consider the homogeneous space $X = G/\Gamma$. In homogeneous dynamics, one studies the natural translation action of another subgroup $H \subset G$ on X, examines properties of orbits and invariant measures of the action, and investigates the asymptotic behavior of the action when elements in H tend to infinity. This kind of dynamical systems gives rise to several classical examples, including geodesic flows and horocycle flows on surfaces of constant negative curvature.

By combining methods from dynamical systems and Lie theory, one is able to obtain stronger results in homogeneous dynamics than in more general dynamics. A typical example is Ratner's theorems for unipotent flows [63], which state that if the subgroup H is generated by unipotent elements, then the closures of H-orbits in X are algebraically defined submanifolds, and the H-invariant measures on X can be also classified in an algebraic way. For some other important cases of H, similar statements have also been conjectured and partially proved to be true (see [12, 24, 26, 40, 48]). On the other hand, certain subgroups Γ (for example, $\mathrm{SL}_n(\mathbb{Z})$) carry a large amount of arithmetic information, and the corresponding dynamical systems are closely related to number theory. For example, methods and results from

This paper is the English revised version of the authors' survey [5] (in Chinese). The first author was supported by NSFC grant 11322101. The second author was supported by EPSRC Programme Grant EP/J018260/1 at the University of York.

[1]Throughout this paper, a Lie group refers to a *real* Lie group. It is also important in homogeneous dynamics to consider other topological groups (for example, p-adic Lie groups), which are not discussed here.

homogeneous dynamics have been used to solve several important questions in Diophantine approximation. To list a few:

- Margulis proved the Oppenheim conjecture [43, 44];
- Kleinbock and Margulis proved the Baker-Sprindžuk conjecture [35];
- Einsiedler, Katok and Lindenstrauss made significant progress towards the Littlewood conjecture [25];
- Shah established important properties of Dirichlet improvable vectors [68].

There have also been important applications of homogeneous dynamics to algebraic number theory, analytic number theory, and other fields. See, for example, [27, 50, 51, 56, 70].

For certain important cases, the action of the subgroup H on the space X is ergodic, and hence the points in X with non-dense H-orbits form a set of measure zero. However, the behavior of non-dense orbits reveals the complexity of the system, and is related to number-theoretic questions. As such, several types of non-dense orbits, including bounded orbits, divergent orbits, and orbits whose closures or limit sets miss a given set, have attracted considerable interest in the last three decades.

In this paper, we give a brief survey of some major problems and results concerning bounded orbits of homogeneous dynamics and their relations with Diophantine approximation. The basic setting of homogeneous dynamics is reviewed in Section 2. Then, in Sections 3–5, we discuss properties of bounded orbits according to different types of dynamical systems (namely, unipotent systems, higher rank diagonalizable systems, and rank-one diagonalizable systems) and their relations with problems in Diophantine approximation, including the Oppenheim conjecture, the Littlewood conjecture, and the Schmidt conjecture. For divergent orbits and other types of non-dense orbits, the reader may consult [7, 17, 23, 32, 37, 39] and the references therein.

2. Basic setting of homogeneous dynamics

Let G be a connected Lie group, $\Gamma \subset G$ be a closed subgroup, and endow the homogeneous space $X = G/\Gamma$ with the natural smooth manifold structure. For $g \in G$, the translation action of the cyclic group $\{g^n : n \in \mathbb{Z}\}$ on X gives rise to a discrete dynamical system $\mathbb{Z} \times X \to X$, $(n, x) \mapsto g^n x$. Similarly, for an element ξ in the Lie algebra $\mathfrak{g}$ of G, the one-parameter group $\{\exp(t\xi) : t \in \mathbb{R}\}$ gives rise to a flow $\mathbb{R} \times X \to X$, $(t, x) \mapsto \exp(t\xi)x$. More generally, in homogeneous dynamics, we are interested in the dynamical system induced by the translation action

$$H \times X \to X, \qquad (h, x) \mapsto hx$$

of a general subgroup $H \subset G$, with emphasis on properties of its orbits and invariant measures.

In order to obtain deep dynamical properties, we need to make some restrictions on the choices of the subgroups H and Γ. First, in studying the asymptotic behavior of the H-action, we expect that elements in H can tend to infinity. Thus, we make the following assumption:

(i) H is a noncompact closed subgroup of G (and hence G is also noncompact).

On the other hand, it turns out that most nontrivial dynamical properties of the H-action originate from the disconnectedness of Γ. To simplify the problem, we assume:

(ii) Γ is a discrete subgroup of G.

Moreover, in order to employ methods and results from ergodic theory, we also assume that

(iii) The coset space X of Γ admits a G-invariant Borel probability measure μ_X (which is called the *Haar measure* and is necessarily unique).

A subgroup $\Gamma \subset G$ satisfying conditions (ii) and (iii) is called a *lattice* in G. It is said to be *cocompact* if X is compact, and is *non-cocompact* otherwise. A subset E of X is said to be *bounded* if it has a compact closure. In this paper, we are mainly concerned with problems of bounded H-orbits in X, which are nontrivial only when Γ is non-cocompact.

Let us give two basic examples.

Example 2.1. $\mathbb{Z}^n$ is a cocompact lattice in $\mathbb{R}^n$, and the quotient space $\mathbb{R}^n/\mathbb{Z}^n$ is an n-dimensional torus.

Example 2.2. Let $n \geq 2$. Then $\mathrm{SL}_n(\mathbb{Z})$ is a non-cocompact lattice in $\mathrm{SL}_n(\mathbb{R})$. Throughout this paper, we denote

$$X_n = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}). \tag{2.1}$$

A proof of the existence of the invariant probability measure on X_n can be found in [58]. Let us give the short proof of the noncompactness of X_n below.

Proof of noncompactness of X_n. Suppose to the contrary that X_n is compact. Then there exists a compact subset $K \subset \mathrm{SL}_n(\mathbb{R})$ such that $K \cdot \mathrm{SL}_n(\mathbb{Z}) = \mathrm{SL}_n(\mathbb{R})$. It follows that

$$K \cdot \mathbb{Z}^n = K \cdot (\mathrm{SL}_n(\mathbb{Z}) \cdot \mathbb{Z}^n) = (K \cdot \mathrm{SL}_n(\mathbb{Z})) \cdot \mathbb{Z}^n = \mathrm{SL}_n(\mathbb{R}) \cdot \mathbb{Z}^n = \mathbb{R}^n.$$

Thus, there exist sequences (g_k) in K and (v_k) in $\mathbb{Z}^n \smallsetminus \{0\}$ with $g_k v_k \to 0$. Since K is compact, we may assume that $g_k \to g$. Hence

$$v_k = g_k^{-1}(g_k v_k) \to g^{-1} 0 = 0,$$

which contradicts $v_k \in \mathbb{Z}^n \smallsetminus \{0\}$. $\square$

Regarding boundedness of subsets in X_n, a criterion is given by Mahler [42], which serves as the bridge between homogeneous dynamics and Diophantine approximation.

Theorem 2.3 (Mahler's compactness criterion)**.** *Let $\pi : \mathrm{SL}_n(\mathbb{R}) \to X_n$ be the projection, that is, $\pi(g) = g \cdot \mathrm{SL}_n(\mathbb{Z})$. Then, for a subset Ω of $\mathrm{SL}_n(\mathbb{R})$, the set $\pi(\Omega)$ is bounded in X_n if and only if the origin in $\mathbb{R}^n$ is an isolated point of the set $\{gv : g \in \Omega, v \in \mathbb{Z}^n\}$.*

It is worth remarking that not every Lie group has a lattice. It can be proved that if G has a lattice, then $|\det(\mathrm{Ad}(g))| = 1$ for every $g \in G$, where $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is the adjoint representation. On the other hand, any connected noncompact semisimple Lie group admits both cocompact lattices and non-cocompact lattices, and any lattice in a solvable Lie group (if it exists) must be cocompact. For proofs of these facts and more properties of lattices, see [46, 54, 58, 72].

One of the most fundamental theorems in homogeneous dynamics is Moore's ergodicity theorem [52]. For simple Lie groups[2], it reads as follows:

Theorem 2.4 (Moore's ergodicity theorem for simple Lie groups)**.** *Let G be a connected noncompact simple Lie group with finite center, and $\Gamma \subset G$ be a lattice. Then the translation action of G on the homogeneous space $X = G/\Gamma$ is* strongly mixing *with respect to the Haar measure μ_X, that is, for any measurable subsets $E_1, E_2 \subset X$, we have*

$$\lim_{G \ni g \to \infty} \mu_X(gE_1 \cap E_2) = \mu_X(E_1)\mu_X(E_2).$$

In particular, the translation action on X of any noncompact closed subgroup $H \subset G$ is ergodic*, that is, for any H-invariant measurable subset $E \subset X$, we have*

$$\mu_X(E) = 0 \qquad or \qquad \mu_X(E) = 1.$$

In view of Moore's ergodicity theorem, it follows that if $H \subset G$ is a noncompact closed subgroup, then the H-orbit of almost every point in X is dense. If X is noncompact, then a bounded H-orbit is non-dense, and hence the set

$$\{x \in X : Hx \text{ is bounded}\} \tag{2.2}$$

is of measure zero.

We conclude this section by the following two basic examples.

Example 2.5 (Geodesic flows and horocycle flows on surfaces of constant negative curvature)**.** Let Σ be a complete surface of constant negative curvature with finite volume. Then the unit tangent bundle $T^1\Sigma$ is isomorphic to a homogeneous space $X = \mathrm{SL}_2(\mathbb{R})/\Gamma$, where $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a lattice. The Liouville measure on $T^1\Sigma$ can be identified with the Haar measure on X. Consider the one-parameter subgroups

$$D = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\} \qquad \text{and} \qquad U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

of $\mathrm{SL}_2(\mathbb{R})$. The dynamical systems induced by the translation actions of D and U on X correspond to the geodesic flow and horocycle flow on $T^1\Sigma$,

[2] Moore's ergodicity theorem also holds for irreducible lattices in semisimple Lie groups.

respectively (see [11]). The investigation of these flows from the viewpoint of homogeneous dynamics originated from the work of Gelfand and Fomin [28]. The geodesic flow and the horocycle flow have very different properties. For example, if Σ is noncompact, then a bounded D-orbit can be quite complicated, while every bounded U-orbit is periodic (see [20]).

For more detailed introduction to homogeneous dynamics, the reader may consult [11, 22, 36, 69].

3. Unipotent systems

We examine bounded orbits of homogeneous dynamics according to different types of the acting group H. First, let us introduce the following concepts.

Definition 3.1. Let G be a connected noncompact Lie group.

1. An element $g \in G$ is said to be Ad_G-*unipotent* if the linear transformation $\mathrm{Ad}(g) - \mathrm{id}_{\mathfrak{g}}$ on $\mathfrak{g}$ is nilpotent, and is Ad_G-*diagonalizable* if $\mathrm{Ad}(g)$ is diagonalizable.
2. Let $\Gamma \subset G$ be a lattice, $X = G/\Gamma$, and $H \subset G$ be a connected noncompact closed subgroup. The dynamical system induced by the translation action of H on X is called a *unipotent system* if H is generated by Ad_G-unipotent elements, and is a *diagonalizable system* if H consists of Ad_G-diagonalizable elements.

Unipotent systems and diagonalizable systems are the most important types of homogeneous dynamics. In a certain sense, the study of the action of a general subgroup H can be reduced to these two cases. In Example 2.5, the horocycle flow is a unipotent system, while the geodesic flow is a diagonalizable system.

In this section, we discuss bounded orbits of unipotent systems. It should be pointed out that if H is noncompact simple, then it follows from the Iwasawa decomposition that H is generated by Ad_G-unipotent elements. Thus, unipotent systems include the case that $H \subset G$ is a noncompact simple Lie subgroup.

First, let us consider the case where $G = \mathrm{SL}_3(\mathbb{R})$, $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, and $X = X_3$ (see (2.1)). Let $H = \mathrm{SO}^+(2,1)$ be the identity component of the orthogonal group of signature $(2,1)$, which is noncompact and simple. In the 1980s, Margulis [43, 44] proved the following result.

Theorem 3.2. *Every bounded* $\mathrm{SO}^+(2,1)$*-orbit in* X_3 *is compact.*

The background of Theorem 3.2 is the Oppenheim conjecture in Diophantine approximation. Recall that Meyer's theorem states that if Q is a nondegenerate indefinite rational quadratic form on $\mathbb{R}^n$, and if $n \geq 5$, then $Q(v) = 0$ for some $v \in \mathbb{Z}^n \smallsetminus \{0\}$ (see [67]). Oppenheim [57] conjectured that for irrational quadratic forms, Meyer's theorem remains valid in the sense of approximation. The conjecture was later strengthened to $n \geq 3$, and can be states as follows:

Conjecture 3.3 (Oppenheim conjecture)**.** *Let $n \geq 3$, Q be a nondegenerate indefinite quadratic form on $\mathbb{R}^n$. Assume that Q is not a constant multiple of a rational quadratic form. Then, for any $\epsilon > 0$, there exists $v \in \mathbb{Z}^n \smallsetminus \{0\}$ such that $|Q(v)| < \epsilon$.*[3]

It is easy to see that if the conjecture holds for some $n = n_0$, then it holds for every $n \geq n_0$. Thus, it suffices to prove the $n = 3$ case. It was observed by Cassels and Swinnerton-Dyer [16] (in an implicit form), and independently by Raghunathan (unpublished), that the $n = 3$ case of the conjecture is equivalent to the statement in Theorem 3.2. Thus Theorem 3.2 implies the truth of the Oppenheim conjecture. Before the appearance of Margulis' dynamical proof, the conjecture was proved for $n \geq 21$ using methods from analytic number theory. Margulis' proof relies on the fact that the group $\mathrm{SO}^+(2,1)$ is generated by unipotent elements as well as dynamical properties of actions of unipotent one-parameter subgroups. It should be noted that so far there is no number-theoretic proof of the full Oppenheim conjecture. For quantitative and effective results on the conjecture, see [29, 41, 47] and the references therein.

We now briefly explain the relation between Theorem 3.2 and the Oppenheim conjecture. Consider the quadratic form on $\mathbb{R}^3$ given by

$$Q_0(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2.$$

Every nondegenerate indefinite quadratic form Q on $\mathbb{R}^3$ is of the form $Q = c(Q_0 \circ g)$, where c is a nonzero constant and $g \in \mathrm{SL}_3(\mathbb{R})$. The equivalence of the Oppenheim conjecture and Theorem 3.2 follows from the following proposition.

Proposition 3.4. *Let $\pi : \mathrm{SL}_3(\mathbb{R}) \to X_3$ be the projection. Then the following statements hold.*

(1) *The orbit $\mathrm{SO}^+(2,1) \cdot \pi(g)$ is bounded if and only if*

$$\inf_{v \in \mathbb{Z}^3 \smallsetminus \{0\}} |Q(v)| > 0.$$

(2) *The orbit $\mathrm{SO}^+(2,1) \cdot \pi(g)$ is closed if and only if Q is a constant multiple of a rational quadratic form.*

To demonstrate the role played by Mahler's compactness criterion as the bridge, let us sketch the proof of (1).

Proof of Proposition 3.4 (1). For any $a \in \mathbb{R}$, the group $H = \mathrm{SO}^+(2,1)$ acts transitively on every connected component of the surface

$$\{v \in \mathbb{R}^3 \smallsetminus \{0\} : Q_0(v) = a\}.$$

[3]The condition $n \geq 3$ is necessary: it is easy to verify that $|x_1^2 - (1+\sqrt{2})^2 x_2^2| \geq 1$ for any $(x_1, x_2) \in \mathbb{Z}^2 \smallsetminus \{0\}$.

It follows that for $v \in \mathbb{R}^3$, $|Q_0(v)|$ is small if and only if $\inf_{h\in H} \|hv\|$ is small. Thus,

$$\begin{aligned}\inf_{v\in\mathbb{Z}^3\smallsetminus\{0\}} |Q(v)| = 0 \quad &\iff \quad \inf_{v\in\mathbb{Z}^3\smallsetminus\{0\}} |Q_0(gv)| = 0 \\ &\iff \quad \inf_{v\in\mathbb{Z}^3\smallsetminus\{0\}} \inf_{h\in H} \|hgv\| = 0 \\ &\iff \quad 0 \in \overline{\{hgv : h \in H, v \in \mathbb{Z}^3 \smallsetminus \{0\}\}}.\end{aligned}$$

In view of Mahler's compactness criterion, the last condition is equivalent to the unboundedness of $H\pi(g)$. □

Remark 3.5. Proposition 3.4 remains true in higher dimensions. In view of Meyer's theorem, it follows that if $p, q > 0$ and $n = p + q \geq 5$, then every $\mathrm{SO}^+(p, q)$-orbit in X_n is unbounded.

In the 1990s, Ratner [59, 60, 61, 62] proved a group of remarkable theorems concerning unipotent systems, including the measure classification theorem, the orbit closure theorem, and the equidistribution theorem (see also [49]). Her orbit closure theorem is as follows:

Theorem 3.6 (Ratner's orbit closure theorem)**.** *Let G be a connected noncompact Lie group, $\Gamma \subset G$ be a lattice, $X = G/\Gamma$, and $H \subset G$ be a connected noncompact closed subgroup. Assume that H is generated by Ad_G-unipotent elements. Then, for every $x \in X$, there exists a connected closed subgroup L of G containing H such that $\overline{Hx} = Lx$.*

It is easy to deduce Theorem 3.2 from Theorem 3.6. On the other hand, Ratner's work implies the following result.

Theorem 3.7. *Under the conditions of Theorem 3.6, if moreover X is noncompact, then the set* (2.2) *is contained in a countable union of proper submanifolds of X, and hence its Hausdorff dimension is less than* $\dim X$*.*

Ratner's work provides a relatively complete qualitative understanding of unipotent systems (see [21, 45, 53, 63]). However, as far as the quantitative and effective aspects are concerned, there are still many open problems. See, for example, [40, 48].

4. Higher rank diagonalizable systems

Let us now consider diagonalizable systems, that is, the case where the acting group H consists of Ad_G-diagonalizable elements. A diagonalizable system is said to be of *rank-one* if $\dim H = 1$, and is of *higher rank* if $\dim H > 1$. These two types of systems exhibit quite different features. In this section we are concerned with higher rank diagonalizable systems. For such systems, one hopes to establish similar results as in the unipotent case. However, this project is far from complete. Most progress made so far is in the case where H is a maximal $\mathbb{R}$-split torus in G.

To avoid technicalities, let us concentrate on the case where $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, and H is the subgroup

$$A := \{\mathrm{diag}(e^{t_1}, \dots, e^{t_n}) : t_i \in \mathbb{R}, t_1 + \cdots + t_n = 0\}.$$

Note that $\dim A = n - 1$, and thus the system is of higher rank if $n \geq 3$. Regarding bounded A-orbits, Margulis [48] conjectured that a statement similar to Theorem 3.2 holds, that is:

Conjecture 4.1. *If $n \geq 3$, then every bounded A-orbit in X_n is compact.*

In the language of Diophantine approximation, a statement equivalent to the $n = 3$ case of Conjecture 4.1 also appeared in the work of Cassels and Swinnerton-Dyer [16] (see Conjecture 4.6 below). So far, the most significant progress towards Conjecture 4.1 is made by Einsiedler, Katok and Lindenstrauss [25]. By studying A-invariant measures on X_n, they proved the following result.

Theorem 4.2. *If $n \geq 3$, then the set*

$$\{x \in X_n : Ax \text{ is bounded}\}$$

has Hausdorff dimension $n - 1$.

Note that there exist countably infinitely many compact A-orbits in X_n. Thus, the set

$$\{x \in X_n : Ax \text{ is compact}\}$$

also has Hausdorff dimension $n - 1$. The following stronger result is also proved in [25].

Theorem 4.3. *Let $n \geq 3$, $A^+ \subset A$ be a subsemigroup with an interior point $g \in A^+$. Consider the subgroup*

$$U(g) = \{u \in \mathrm{SL}_n(\mathbb{R}) : \lim_{k \to +\infty} g^{-k} u g^k = I_n\}$$

of $\mathrm{SL}_n(\mathbb{R})$. Then for every $x \in X_n$, the set

$$\{u \in U(g) : A^+ ux \text{ is bounded}\}$$

has Hausdorff dimension 0.

An equivalent formulation of Conjecture 4.1 can be given using the notion of a minimal set. Recall that for a subgroup $S \subset \mathrm{SL}_n(\mathbb{R})$, a nonempty subset $Y \subset X$ is *S-minimal* if it is closed, S-invariant, and contains no nonempty proper S-invariant closed subset. The following result is proved in [8].

Theorem 4.4. *Let $n \geq 3$, and let*

$$A_{ij} = \{\mathrm{diag}(a_1, \dots, a_n) \in A : a_i = a_j\}, \qquad 1 \leq i < j \leq n.$$

Then the following statements are equivalent.

(1) *Conjecture 4.1 holds for n.*
(2) *Any compact A-minimal set in X_n is A_{ij}-minimal for every A_{ij}.*

Conjecture 4.1 and Theorem 4.3 are closely related to the Littlewood conjecture in Diophantine approximation. For $a \in \mathbb{R}$, let $\langle a \rangle$ denote the distance of a to the nearest integer, that is, $\langle a \rangle = \inf_{k\in\mathbb{Z}} |a-k|$. Littlewood proposed the following conjecture around 1930.

Conjecture 4.5 (Littlewood conjecture)**.** *For any $a, b \in \mathbb{R}$, we have*

$$\inf_{q\in\mathbb{N}} q\langle qa\rangle\langle qb\rangle = 0.$$

The Littlewood conjecture is one of the most important open problems in Diophantine approximation. It is proved in [16] that the following conjecture implies the Littlewood conjecture.

Conjecture 4.6. *Let f_1, f_2, f_3 be linearly independent linear forms on $\mathbb{R}^3$, and let $F = f_1 f_2 f_3$. Assume that F is not a constant multiple of a rational polynomial. Then, for any $\epsilon > 0$, there exists $v \in \mathbb{Z}^3 \smallsetminus \{0\}$ such that $|F(v)| < \epsilon$.*

Conjecture 4.6 is comparable in spirit to the Oppenheim conjecture. Similar to the equivalence of Theorem 3.2 and the Oppenheim conjecture, it can be proved using Mahler's compactness criterion that the $n = 3$ case of Conjecture 4.1 is equivalent to Conjecture 4.6. Thus Conjecture 4.1 implies the Littlewood conjecture.

On the other hand, using Theorem 4.3, it is proved in [25] that the Littlewood conjecture holds up to a set of Hausdorff dimension zero:

Theorem 4.7. *The set*

$$\{(a,b) \in \mathbb{R}^2 : \inf_{q\in\mathbb{N}} q\langle qa\rangle\langle qb\rangle > 0\}$$

has Hausdorff dimension 0*.*

To explain the relation between Theorem 4.3 and the Littlewood conjecture, let us consider the subsemigroup

$$A^+ = \{\operatorname{diag}(e^{t_1}, e^{t_2}, e^{-(t_1+t_2)}) : t_1, t_2 \geq 0\} \tag{4.1}$$

of $\mathrm{SL}_3(\mathbb{R})$. For $a, b \in \mathbb{R}$, denote

$$u_{a,b} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{a,b} = u_{a,b} \cdot \mathrm{SL}_3(\mathbb{Z}).$$

The following statement can be proved using Mahler's compactness criterion.

Proposition 4.8. *The orbit $A^+ x_{a,b}$ in X_3 is bounded if and only if*

$$\inf_{q\in\mathbb{N}} q\langle qa\rangle\langle qb\rangle > 0.$$

By using Proposition 4.8, it is easy to deduce Theorem 4.7 from Theorem 4.3:

Proof of Theorem 4.7. Take $n = 3$ in Theorem 4.3, and let A^+ be the subsemigroup given by (4.1). Then $g = \operatorname{diag}(2, 2, 1/4)$ is an interior point of A^+, and we have $U(g) = \{u_{a,b} : a, b \in \mathbb{R}\}$. It follows that the set $\{(a, b) \in \mathbb{R}^2 : A^+x_{a,b} \text{ is bounded}\}$ has Hausdorff dimension 0. Now Theorem 4.7 follows from Proposition 4.8. $\square$

5. Rank-one diagonalizable systems

This section is devoted to bounded orbits of rank-one diagonalizable systems. In this case, it is customary to rewrite the subgroup H as F. The following statement was first conjectured by Margulis [45], and was later proved by Kleinbock and Margulis [34] (see also [37]).

Theorem 5.1. *Let G be a noncompact Lie group, $\Gamma \subset G$ be a non-cocompact lattice, $X = G/\Gamma$, and F be an Ad_G-diagonalizable one-parameter subgroup of G. Then the set*

$$\{x \in X : Fx \text{ is bounded}\} \tag{5.1}$$

has Hausdorff dimension equal to $\dim X$*.*

Comparing Theorem 5.1 with Theorems 3.7 and 4.2, a difference can be seen between rank-one diagonalizable systems and the systems considered in the previous two sections. It is conjectured in [6] that a statement stronger than Theorem 5.1 should be true. To state the conjecture, let us first review the notion of a winning set for Schmidt's game [64].

Definition 5.2 (Schmidt's game and its winning sets)**.** Let X be a complete metric space, $S \subset X$ be a subset, and $\alpha, \beta \in (0, 1)$ be real numbers. Two players, Alice and Bob, play a game on X as follows: Bob starts the game by choosing a closed ball $B_0 \subset X$. After B_i is chosen ($i \geq 0$), Alice chooses a closed ball $A_i \subset B_i$ of radius α times the radius of B_i, and then Bob chooses a closed ball $B_{i+1} \subset A_i$ of radius β times the radius of A_i. This gives a nested sequence

$$B_0 \supset A_0 \supset B_1 \supset A_1 \supset \cdots$$

of closed balls. The intersection of all these balls is a one-point set $\{x_\infty\}$. The rule of the game says that *Alice wins* if $x_\infty \in S$, and *Bob wins* otherwise. The set S is (α, β)*-winning* if Alice has a winning strategy, is α*-winning* if it is (α, β)-winning for any $\beta \in (0, 1)$, and is *winning* if it is α-winning for some $\alpha \in (0, 1)$.

It is easy to see that if S is winning, then it must be dense in X. The following properties are proved in [64]:

Proposition 5.3. *Let X be a complete metric space.*

1. *For any $\alpha \in (0, 1)$, a countable intersection of α-winning sets is α-winning.*
2. *If X is a Riemannian manifold, then a winning subset of X has Hausdorff dimension equal to* $\dim X$*.*

The conjecture proposed in [6] is as follows:

Conjecture 5.4. *Let G, Γ and X be as in Theorem 5.1, and endow X with the Riemannian metric induced from a right-invariant Riemannian metric on G. Then there exists $\alpha \in (0,1)$ such that the set (5.1) is α-winning*[4] *for every Ad_G-diagonalizable one-parameter subgroup F of G.*

In view of Proposition 5.3, the truth of Conjecture 5.4 will imply that for any countable family $\{F_k\}$ of Ad_G-diagonalizable one-parameter subgroups of G, the set

$$\{x \in X : \text{all } F_k x \text{ are bounded}\}$$

has Hausdorff dimension equal to $\dim X$. Some partial results on Conjecture 5.4 are as follows:

Theorem 5.5. *The following statements hold.*

(1) ([19]) *Conjecture 5.4 holds if G is semisimple of $\mathbb{R}$-rank 1.*
(2) ([6]) *Conjecture 5.4 holds if $G = \mathrm{SL}_3(\mathbb{R})$ and $\Gamma = \mathrm{SL}_3(\mathbb{Z})$.*
(3) ([4]) *Conjecture 5.4 holds if G is the product of finitely many copies of $\mathrm{SL}_2(\mathbb{R})$.*
(4) ([15, 7]) *If $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, and*
$$F = \{\mathrm{diag}(e^{t/p} I_p, e^{-t/q} I_q) : t \in \mathbb{R}\}, \qquad p + q = n,$$
then the set (5.1) is winning.
(5) ([30]) *If $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, and*
$$F = \{\mathrm{diag}(e^{rt} I_{n-2}, e^{st}, e^{-t}) : t \in \mathbb{R}\}, \qquad r \geq s \geq 0, \quad (n-2)r + s = 1,$$
then the set (5.1) is winning.

The motivation for Conjecture 5.4 is the Schmidt conjecture in Diophantine approximation. Let $d \geq 1$, and denote

$$W_d = \{(r_1, \ldots, r_d) \in \mathbb{R}^d : r_i \geq 0, r_1 + \cdots + r_d = 1\}.$$

For $\mathbf{r} = (r_1, \ldots, r_d) \in W_d$, consider the set

$$\mathrm{Bad}(\mathbf{r}) = \{(a_1, \ldots, a_d) \in \mathbb{R}^d : \inf_{q \in \mathbb{N}} \max_{1 \leq i \leq d} q^{r_i} \langle q a_i \rangle > 0\}.$$

Vectors in $\mathrm{Bad}(\mathbf{r})$ are said to be *badly approximable with weight* $\mathbf{r}$. The set $\mathrm{Bad}(\mathbf{r})$ is a fundamental object of study in Diophantine approximation. When $d = 1$, it reduces to the set of badly approximable numbers. Schmidt [64, 65] proved that $\mathrm{Bad}(\frac{1}{d}, \ldots, \frac{1}{d})$ is a winning set, and proposed the following conjecture in [66]:

Conjecture 5.6 (Schmidt conjecture)**.** *For $d = 2$, we have*

$$\mathrm{Bad}(\tfrac{1}{3}, \tfrac{2}{3}) \cap \mathrm{Bad}(\tfrac{2}{3}, \tfrac{1}{3}) \neq \varnothing.$$

[4] The conjecture in [6] is in fact stronger: it states that the set (5.1) is hyperplane absolute winning (HAW). The notion of a HAW set is introduced in [14, 38] and has better properties. Here we only discuss winning sets in the sense of Definition 5.2 for simplicity.

The Schmidt conjecture was proved by Badziahin, Pollington and Velani [9]. They also proved that for any countable subset $\mathcal{R} \subset W_2$ satisfying a certain technical condition (which holds automatically if $\mathcal{R}$ is finite), the intersection $\bigcap_{\mathbf{r}\in\mathcal{R}} \mathrm{Bad}(\mathbf{r})$ has Hausdorff dimension 2. Subsequently, this technical condition was removed in [1], and then the following result was proved in [2]:

Theorem 5.7. *There exists $\alpha \in (0,1)$ such that* $\mathrm{Bad}(\mathbf{r})$ *is α-winning for any $\mathbf{r} \in W_2$.*

In view of Proposition 5.3, this gives another proof of the Schmidt conjecture. See also [3, 10, 55] for further results. In dimension $d \geq 3$, Beresnevich [13] proved that $\bigcap_{\mathbf{r}\in\mathcal{R}} \mathrm{Bad}(\mathbf{r})$ has Hausdorff dimension d for any countable subset $\mathcal{R} \subset W_d$ satisfying a technical condition similar to that in [9], and the condition was then removed by Yang [71]. Moreover, it is proved in [31] that $\mathrm{Bad}(\mathbf{r})$ is winning if the weight $\mathbf{r} = (r_1, \ldots, r_d)$ satisfies $r_1 = \cdots = r_{d-1} \geq r_d$. For a general weight $\mathbf{r} \in W_d$, whether $\mathrm{Bad}(\mathbf{r})$ is a winning set remains a challenging open problem.

Let us now explain the relation between badly approximable vectors and bounded orbits. Take $n = d+1$. For a weight $\mathbf{r} = (r_1, \ldots, r_d) \in W_d$, consider the one-parameter subsemigroup

$$F_{\mathbf{r}}^+ = \{\mathrm{diag}(e^{r_1 t}, \ldots, e^{r_d t}, e^{-t}) : t \geq 0\}$$

of $\mathrm{SL}_n(\mathbb{R})$. For a row vector $\mathbf{a} \in \mathbb{R}^d$, let

$$x_{\mathbf{a}} = \begin{pmatrix} I_d & \mathbf{a}^{\mathrm{T}} \\ 0 & 1 \end{pmatrix} \mathrm{SL}_n(\mathbb{Z}) \in X_n.$$

Similar to Proposition 4.8, the following result was proved by Dani [18] and Kleinbock [33] using Mahler's compactness criterion.

Proposition 5.8 (Dani-Kleinbock correspondence)**.** *The orbit $F_{\mathbf{r}}^+ x_{\mathbf{a}}$ is bounded if and only if $\mathbf{a} \in \mathrm{Bad}(\mathbf{r})$.*

Consider the 2-dimensional torus $\mathbb{T}^2 = \{x_{\mathbf{a}} : \mathbf{a} \in \mathbb{R}^2\}$ embedded in X_3. In view of the Dani-Kleinbock correspondence, the Schmidt conjecture is equivalent to the statement that there exists $x \in \mathbb{T}^2$ such that the orbits $F_{(\frac{1}{3},\frac{2}{3})}^+ x$ and $F_{(\frac{2}{3},\frac{1}{3})}^+ x$ are both bounded. Similarly, Theorem 5.7 can be restated as the following dynamical statement: For any one-parameter subsemigroup F^+ of the semigroup A^+ given in (4.1), the set

$$\{x \in \mathbb{T}^2 : F^+ x \text{ is bounded}\}$$

is a winning subset of $\mathbb{T}^2$. Therefore, Conjecture 5.4 can be viewed as a dynamical Schmidt conjecture. Finally, we point out that a similar conjecture for the so-called expanding horospherical subgroups is also formulated in [6]. To avoid technicalities, we invite the reader to consult [6] directly.

References

[1] An J. Badziahin-Pollington-Velani's theorem and Schmidt's game. Bull Lond Math Soc, 2013, 45: 721–733

[2] An J. 2-dimensional badly approximable vectors and Schmidt's game. Duke Math J, 2016, 165: 267–284

[3] An J, Beresnevich V, Velani S. Badly approximable points on planar curves and winning. Adv Math, 2018, 324: 148–202

[4] An J, Ghosh A, Guan L, Ly T. Bounded orbits of diagonalizable flows on finite volume quotients of products of $\mathrm{SL}_2(\mathbb{R})$. ArXiv:1605.08510, 2016

[5] An J, Guan L. Bounded orbits of homogeneous dynamics (in Chinese). Sci Sin Math, 2017, 47: 1623–1634

[6] An J, Guan L, Kleinbock D. Bounded orbits of diagonalizable flows on $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$. Int Math Res Not, 2015, 2015: 13623–13652

[7] An J, Guan L, Kleinbock D. Nondense orbits and generalized indefinite binary forms. In preparation

[8] An J, Weiss B. Remarks on minimal sets and conjectures of Cassels, Swinnerton-Dyer, and Margulis. Mosc J Comb Number Theory, 2013, 3: 24–43

[9] Badziahin D, Pollington A, Velani S. On a problem in simultaneous Diophantine approximation: Schmidt's conjecture. Ann of Math, 2011, 174: 1837–1883

[10] Badziahin D, Velani S. Badly approximable points on planar curves and a problem of Davenport. Math Ann, 2014, 359: 969–1023

[11] Bekka M, Mayer M. Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces. Cambridge: Cambridge University Press, 2000

[12] Benoist Y. Recurrence on the space of lattices. In: Proceedings of the International Congress of Mathematicians (Seoul, 2014), vol. III. Seoul: Kyung Moon Sa Co Ltd, 2014, 11–25

[13] Beresnevich V. Badly approximable points on manifolds. Invent Math, 2015, 202: 1199–1240

[14] Broderick R, Fishman L, Kleinbock D, Reich A, Weiss B. The set of badly approximable vectors is strongly C^1 incompressible. Math Proc Cambridge Philos Soc, 2012, 153: 319–339

[15] Broderick R, Fishman L, Simmons D. Badly approximable systems of affine forms and incompressibility on fractals. J Number Theory, 2013, 133: 2186–2205

[16] Cassels J, Swinnerton-Dyer H. On the product of three homogeneous linear forms and the indefinite ternary quadratic forms. Philos Trans Roy Soc London Ser A, 1955, 248: 73–96

[17] Cheung Y, Chevallier N. Hausdorff dimension of singular vectors. Duke Math J, 2016, 165: 2273–2329

[18] Dani S. Divergent trajectories of flows on homogeneous spaces and Diophantine approximation. J Reine Angew Math, 1985, 359: 55–89

[19] Dani S. Bounded orbits of flows on homogeneous spaces. Comment Math Helv, 1986, 61: 636–660

[20] Dani S. On badly approximable numbers, Schmidt games and bounded orbits of flows. In: Number Theory and Dynamical Systems (York, 1987). Cambridge: Cambridge University Press, 1989, 69–86

[21] Dani S. Flows on homogeneous spaces and Diophantine approximation. In: Proceedings of the International Congress of Mathematicians (Zürich, 1994), vol II. Basel: Birkhäuser, 1995, 780–789

[22] Dani S. Dynamical systems on homogeneous spaces. In: Dynamical Systems, Ergodic Theory and Applications, 2nd ed. Encyclopaedia of Mathematical Sciences, vol. 100. Berlin: Springer-Verlag, 2000, 264–359

[23] Das T, Fishman L, Simmons D, Urbański M. A variational principle in the parametric geometry of numbers, with applications to metric Diophantine approximation. C R Math Acad Sci Paris, 2017, 355: 835–846
[24] Einsiedler M. Applications of measure rigidity of diagonal actions. In: Proceedings of the International Congress of Mathematicians (Hyderabad, 2010), vol. III. New Delhi: Hindustan Book Agency, 2010, 1740–1759
[25] Einsiedler M, Katok A, Lindenstrauss L. Invariant measures and the set of exceptions to Littlewood's conjecture. Ann of Math, 2006, 164: 513–560
[26] Einsiedler M, Lindenstrauss E. Diagonalizable flows on locally homogeneous spaces and number theory. In: Proceedings of the International Congress of Mathematicians (Madrid, 2006), vol. II. Zürich: Eur Math Soc, 2006, 1731–1759
[27] Eskin A. Counting problems and semisimple groups. In: Proceedings of the International Congress of Mathematicians (Berlin, 1998), vol. II. Bielefeld: Doc Math, 1998, 539–552
[28] Gelfand I, Fomin S. Geodesic flows on manifolds of constant negative curvature. Uspehi Matem Nauk, 1952, 7: 118–137
[29] Götze F, Margulis G. Distribution of values of quadratic forms at integral points. ArXiv:1004.5123, 2010
[30] Guan L, Wu W. Bounded orbits of certain diagonalizable flows on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$. Trans Amer Math Soc, in press
[31] Guan L, Yu J. Weighted badly approximable vectors and games. Int Math Res Not, in press
[32] Kadyrov S, Kleinbock D, Lindenstrauss E, Margulis G. Singular systems of linear forms and non-escape of mass in the space of lattices. J Anal Math, 2017, 133: 253–277
[33] Kleinbock D. Flows on homogeneous spaces and Diophantine properties of matrices. Duke Math J, 1998, 95: 107–124
[34] Kleinbock D, Margulis G. Bounded orbits of nonquasiunipotent flows on homogeneous spaces. Amer Math Soc Transl, 1996, 171: 141–172
[35] Kleinbock D, Margulis G. Flows on homogeneous spaces and Diophantine approximation on manifolds. Ann of Math, 1998, 148: 339–360
[36] Kleinbock D, Shah N, Starkov A. Dynamics of subgroup actions on homogeneous spaces of Lie groups and applications to number theory. In: Handbook of Dynamical Systems, vol. 1A. Amsterdam: North-Holland, 2002, 813–930
[37] Kleinbock D, Weiss B. Modified Schmidt games and a conjecture of Margulis. J Mod Dyn, 2013, 7: 429–460
[38] Kleinbock D, Weiss B. Values of binary quadratic forms at integer points and Schmidt games. In: Recent Trends in Ergodic Theory and Dynamical Systems. Contemp Math, vol. 631. Providence, RI: Amer Math Soc, 2015, 77–92
[39] Liao L, Shi R, Solan O, Tamam N. Hausdorff dimension of weighted singular vectors. ArXiv:1605.01287, 2016
[40] Lindenstrauss E. Equidistribution in homogeneous spaces and number theory. In: Proceedings of the International Congress of Mathematicians (Hyderabad, 2010), vol. I. New Delhi: Hindustan Book Agency, 2010, 531–557
[41] Lindenstrauss E, Margulis G. Effective estimates on indefinite ternary forms. Israel J Math, 2014, 203: 445–499
[42] Mahler K. On lattice points in n-dimensional star bodies I: existence theorems. Proc Roy Soc London Ser A, 1946, 187: 151–187
[43] Margulis G. Indefinite quadratic forms and unipotent flows on homogeneous spaces. In: Dynamical Systems and Ergodic Theory (Warsaw, 1986). Banach Center Publ, vol. 23. Warsaw: Polish Scientific Publishers PWN, 1989, 399–409

[44] Margulis G. Discrete subgroups and ergodic theory. In: Number Theory, Trace Formulas and Discrete Groups (Oslo, 1987). Boston, MA: Academic Press, 1989, 377–398

[45] Margulis G. Dynamical and ergodic properties of subgroup actions on homogeneous spaces with applications to number theory. In: Proceedings of the International Congress of Mathematicians (Kyoto, 1990), vol. I. Tokyo: Math Soc Japan, 1991, 193–215

[46] Margulis G. Discrete Subgroups of Semisimple Lie Groups. Berlin: Springer-Verlag, 1991

[47] Margulis G. Oppenheim conjecture. In: Fields Medallists' Lectures. World Sci Ser 20th Century Math, vol. 5. River Edge, NJ: World Sci Publ, 1997, 272–327

[48] Margulis G. Problems and conjectures in rigidity theory. In: Mathematics: Frontiers and Perspectives. Providence, RI: Amer Math Soc, 2000, 161–174

[49] Margulis G, Tomanov G. Invariant measures for actions of unipotent groups over local fields on homogeneous spaces. Invent Math, 1994, 116: 347–392.

[50] Marklof J. The low-density limit of the Lorentz gas: periodic, aperiodic and random. In: Proceedings of the International Congress of Mathematicians (Seoul, 2014), vol. III. Seoul: Kyung Moon Sa Co Ltd, 2014, 623–646

[51] Michel P, Venkatesh A. Equidistribution, L-functions and ergodic theory: on some problems of Yu. Linnik. In: Proceedings of the International Congress of Mathematicians (Madrid, 2006), vol. II. Zürich: Eur Math Soc, 2006, 421–457

[52] Moore C. Ergodicity of flows on homogeneous spaces. Amer J Math, 1966, 88: 154–178

[53] Morris, D. Ratner's Theorems on Unipotent Flows. Chicago, IL: University of Chicago Press, 2005

[54] Morris, D. Introduction to Arithmetic Groups. Deductive Press, 2015

[55] Nesharim E, Simmons D. Bad(s, t) is hyperplane absolute winning. Acta Arith, 2014, 164: 145–152

[56] Oh H. Dynamics on geometrically finite hyperbolic manifolds with applications to Apollonian circle packings and beyond. In: Proceedings of the International Congress of Mathematicians (Hyderabad, 2010), vol. III. New Delhi: Hindustan Book Agency, 2010, 1308–1331

[57] Oppenheim A. The minima of indefinite quaternary quadratic forms. Proc Nat Acad Sci USA, 1929, 15: 724–727

[58] Raghunathan M. Discrete Subgroups of Lie Groups. New York-Heidelberg: Springer-Verlag, 1972

[59] Ratner M. Strict measure rigidity for unipotent subgroups of solvable groups. Invent Math, 1990, 101: 449–482

[60] Ratner M. On measure rigidity of unipotent subgroups of semisimple groups. Acta Math, 1990, 165: 229–309

[61] Ratner M. On Raghunathan's measure conjecture. Ann of Math, 1991, 134: 545–607

[62] Ratner M. Raghunathan's topological conjecture and distributions of unipotent flows. Duke Math J, 1991, 63: 235–280

[63] Ratner M. Interactions between ergodic theory, Lie groups, and number theory. In: Proceedings of the International Congress of Mathematicians (Zürich, 1994), vol I. Basel: Birkhäuser, 1995, 157–182

[64] Schmidt W. On badly approximable numbers and certain games. Trans Amer Math Soc, 1966, 123: 178–199

[65] Schmidt W. Diophantine Approximation. Lecture Notes in Mathematics, vol. 785. Berlin: Springer, 1980.

[66] Schmidt W. Open problems in Diophantine approximation. In: Diophantine Approximations and Transcendental Numbers (Luminy, 1982). Progr Math, vol. 31. Boston, MA: Birkhäuser, 1983, 271–287

[67] Serre J-P. A Course in Arithmetic. Graduate Texts in Mathematics, vol. 7. New York-Heidelberg: Springer-Verlag, 1973
[68] Shah N. Equidistribution of translates of curves on homogeneous spaces and Dirichlet's approximation. In: Proceedings of the International Congress of Mathematicians (Hyderabad, 2010), vol. III. New Delhi: Hindustan Book Agency, 2010, 1332–1343
[69] Starkov A. Dynamical Systems on Homogeneous Spaces. Providence, RI: American Mathematical Society, 2000
[70] Ullmo E. Théorie ergodique et géométrie arithmétique. In: Proceedings of the International Congress of Mathematicians (Beijing, 2002), vol. II. Beijing: Higher Ed Press, 2002, 197–206
[71] Yang L. Badly approximable points on curves and unipotent orbits in homogeneous spaces. ArXiv:1703.03461, 2017
[72] Zimmer R. Ergodic Theory and Semisimple Groups. Basel: Birkhäuser, 1984

School of Mathematical Sciences, Peking University, Beijing, 100871, China
E-mail address: anjinpeng@gmail.com

Department of Mathematics, University of York, Heslington, York, YO10 5DD, UK
E-mail address: lifan.guan@york.ac.uk