随机游动的常返和相遇问题

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随机游动

- The problem of the random walk. Nature. 72, (1905),294.
- ξ_1，ξ_2, ..., ξ_n 独立同分布随机变量
- $S_n = \xi_1 + \xi_2 + \ldots\ldots + \xi_n$
- 称随机变量序列$\{S_n, n\geq 0\}$为随机游动,
- 群上的随机游动 random walks on groups.
- 平面格点上的简单随机游动.(demo)

- Karl Pearson (1857-1936)

图G= (V, E)上的简单随机游动, 其实是马氏链.

以顶点集V为状态空间的马氏链, 转移概率

$$p(x,y) = \begin{cases} 1/d_x & \text{if } x \sim y \\ 0 & \text{otherwise} \end{cases}$$

其中d_x为顶点x的度数,即与x相连的边数.

更一般的,可以在赋权图G= (V, E, W)上定义马氏链,
以顶点集V为状态空间的, 转移概率

$$p(x,y) = \begin{cases} w_{xy}/w_x & \text{if } x \sim y \\ 0 & \text{otherwise} \end{cases}$$

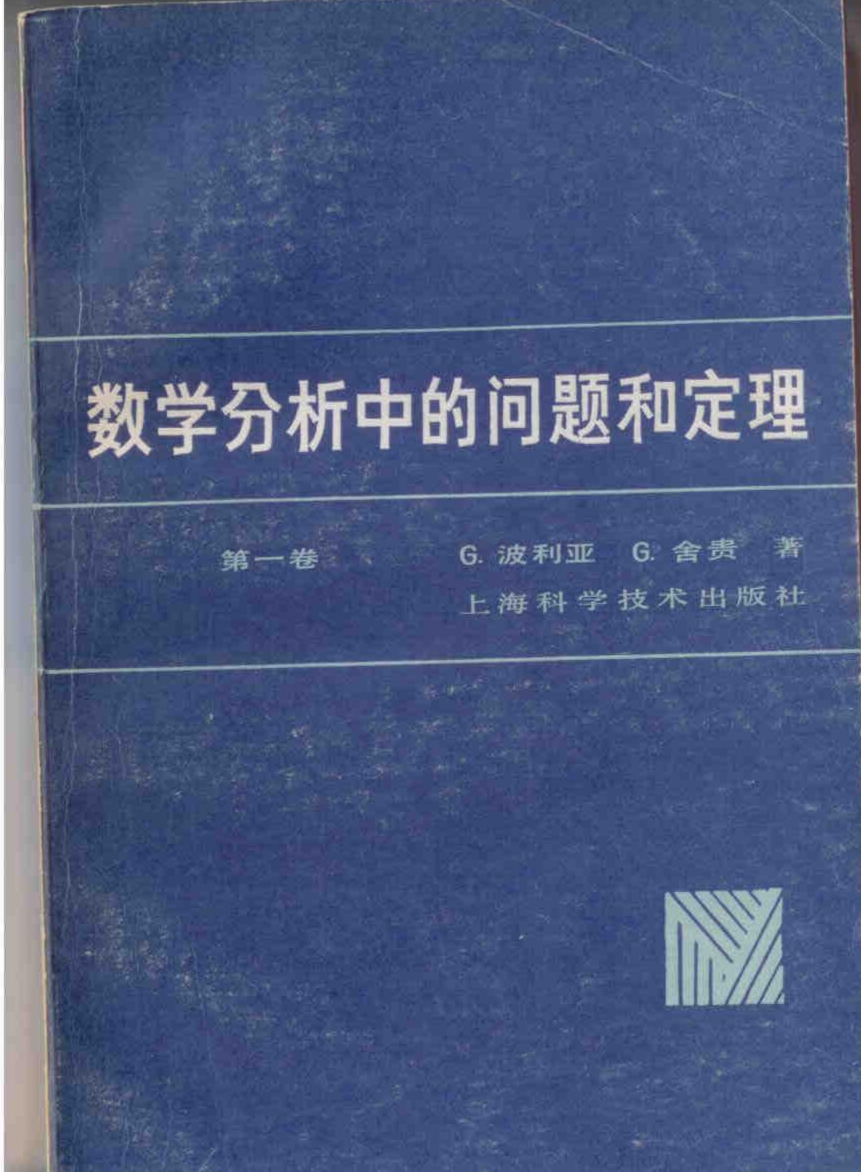
其中$w_x = \Sigma_{x\sim y}\, w_{xy}$, 相应的马氏链是可配称的, 称为图上的随机游动.

George Polya遇到的尴尬事

... he and his fiancée (would) also set out for a stroll in the woods, and then suddenly I met them there. And then I met them the same morning repeatedly. I don't remember how many times, but certainly much too often and I felt embarrassed. It looked as if I was snooping around which was, I assure you, not the case.   I met them by accident -but how likely was it that it happened by accident and not on purpose?

导致他引入常返的概念.

- Polya 1887-1985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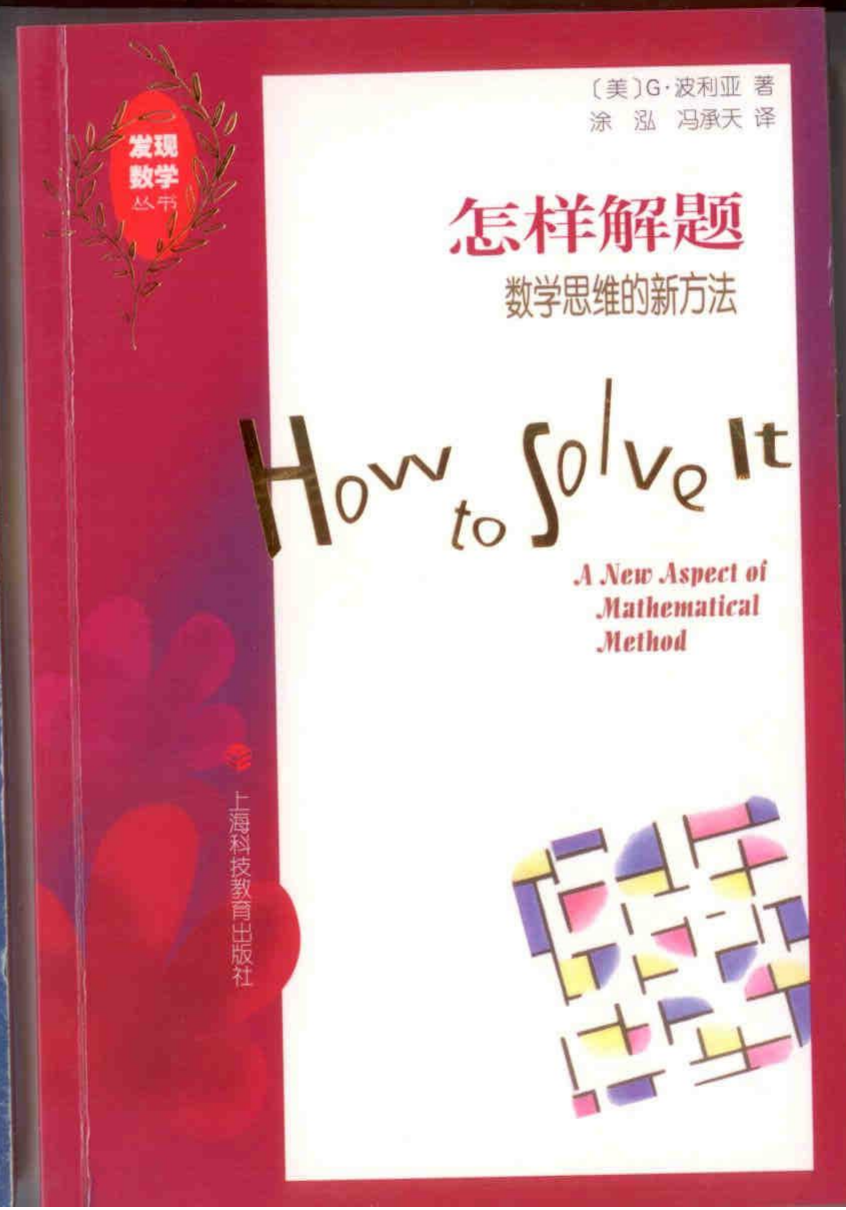
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$X_n = Y_n$ 等价于 $X_n - Y_n = 0$

X_n与 Y_n 经常相遇 $\Leftrightarrow$ $Z_n = X_n - Y_n$ 经常返回到原点

常返成了马氏链需要研究的第一个问题.

几个等价判别准则：(1) $P_x(T_x < \infty) = 1$.
(2) 返回原点无穷多次的概率为1.
(3) 格林函数 $G(x,x) = \Sigma_n\, p_n(x,x) = \infty$

主要结论： 二维随机游动是常返的
而三维随机游动是非常返.

A drunk man will find his way home, but a drunk bird may get lost forever

--Shizuo Kakutani

- 角谷静夫
- (1911-2004)

Feller书上的问题

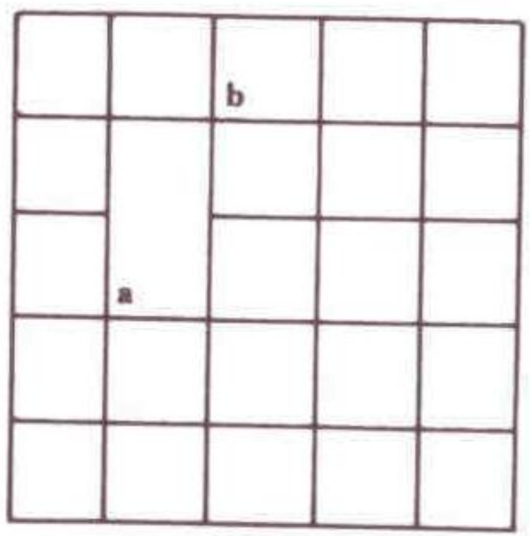
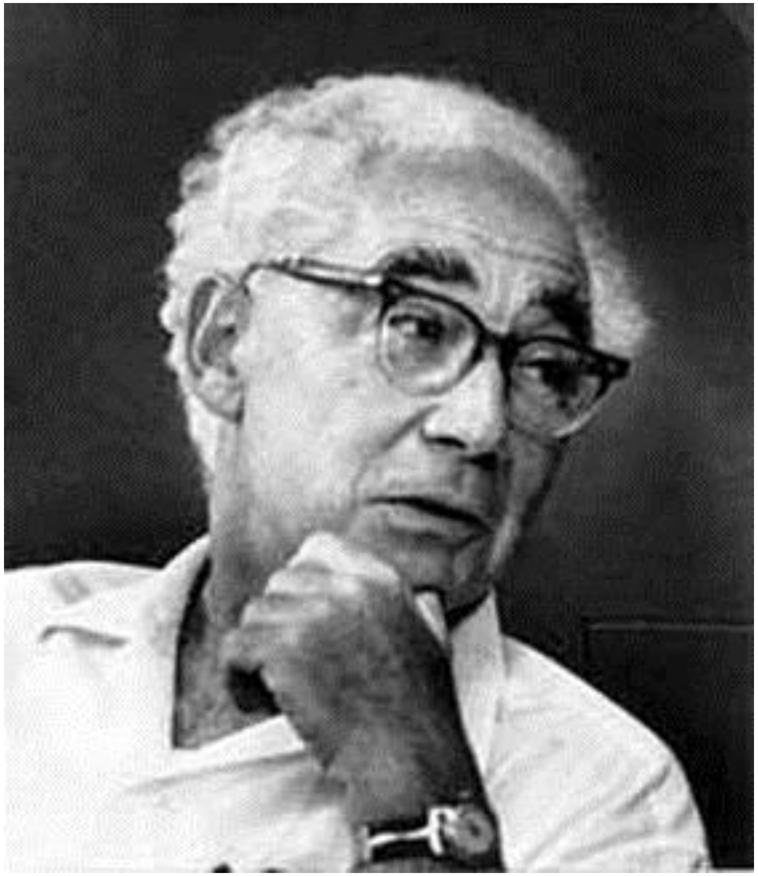


FIGURE 4.2



- William Feller (1906-1970)

原问题: 从原点出发的随机游动是否一定回到原点?

新问题: **随机游动**会不会到达某个点或某个区域?

问题很重要.
若是赌博, 回到原点意味着赌徒输光, 这就变成了赌徒破产问题.
如果是保险公司的保费收入与赔偿, 这意味着公司会不会破产?

用τ_A=随机游动到达区域A的时刻,
τ_x=随机游动到达某点x的时刻.
这是一个不确定的量, 有时用$\tau_A(\omega)$来表示.

例如$A=(-\infty, 0]$为破产区域.

$\tau_A(\omega) < \infty$ 有限时间内破产

$\{\omega : \tau_A(\omega) < \infty\}$ **有限时间内破产**这一事件

人们关心的问题是破产概率有多大?

$$P(\omega : \tau_A(\omega) < \infty) = P(\tau_A < \infty)$$

这与初值有关, 若$X_0=x$,

$$P_x(\tau_A < \infty)$$

原问题 $P_0(\tau_1 < \infty) = 1$?

更一般的问题: $P_x(\tau_A < \tau_B)$.

例如, A= $(-\infty, 0)$为破产区域, B= $[N, \infty)$为赢钱罢赌的区域.

对于公平博弈 (一维简单随机游动)而言

$$P_x(\tau_A < \tau_B) = (1/2)P_{x+1}(\tau_A < \tau_B) + (1/2)\, P_{x-1}(\tau_A < \tau_B).$$

记$P_x(\tau_A < \tau_B) = f(x)$, 那么,
若$y \in A$, 则$f(y) = 1$;
若$z \in B$, 则$f(z) = 0$.
对于一般的x,

$$f(x) = (1/2)\, f(x+1) + (1/2)\, f(x-1).$$

这是线性差分方程组, 并不难解. 譬如 $A = \{0\}$, $B =\{N\}$,

$$f(x) = 1 - (x/N).$$

初始赌资越少, 破产概率越大, 这很公平.

令$N \to \infty$, $\tau_B \to \infty$, $f(x)= P_x(\tau_0 < \infty) \to 1$.
说明一维简单随机游动是常返的.

对于其他情形, 仍可列出方程.
若顶点x有d个邻居$y_1, y_2, \ldots\ldots, y_d$, 则

$$f(x) = [f(y_1) + f(y_2) + \ldots\ldots. + f(y_d)] / d.$$

顶点x的取值是周围顶点取值的算术平均.

这叫调和方程, 见过这样的方程组吗?

电压就满足同样方程!

设A处电压为1, B处电压为0, 则在x处的电压就是f(x).

把A中所有顶点视为一个顶点, B中所有顶点视为一个顶点，进一步设A ∪ B之外的顶点个数是有限的,
R_{AB}为A与B之间的**有效电阻**.
R_{xB}为x与B之间的**有效电阻**.
R_{xA}为x与A之间的**有效电阻**.

若$y \in A$, $f(y) = 1$; $z \in B$, $f(z) = 0$; 则

$$f(x) = 1 - R_{xA}/R_{AB}.$$

可取一列$B_n = B(n)$使得B_n的余集$(B_n)^c$越来越大, 这样
$\tau_{B(n)} \to \infty$.

一般而言, 当$n ---- > \infty$, R_{xA}不会增大, 而$R_{AB(n)}$单调上升.

$R_{A\infty} = \lim_n R_{AB(n)}$ 表示A与∞之间的有效电阻.

$\lim_n f(x) = 1$ **当且仅当** $\lim_n R_{AB(n)} = \infty$.

定理: 随机游动是常返的 <===>
相应的电网络的(从原点到无穷远的)有效电阻是∞.

例 在一维直线上, 从原点到无穷远的电阻为∞.

例 在二维格点上, 从原点到无穷远的电阻为∞.

例在三维格点上, 从原点到无穷远的电阻是有限的.

构造一个能量有限的**流**. 电流的能量最小.

电阻是单调的

平面格点上删除一些边, 随机游动还是常返的.

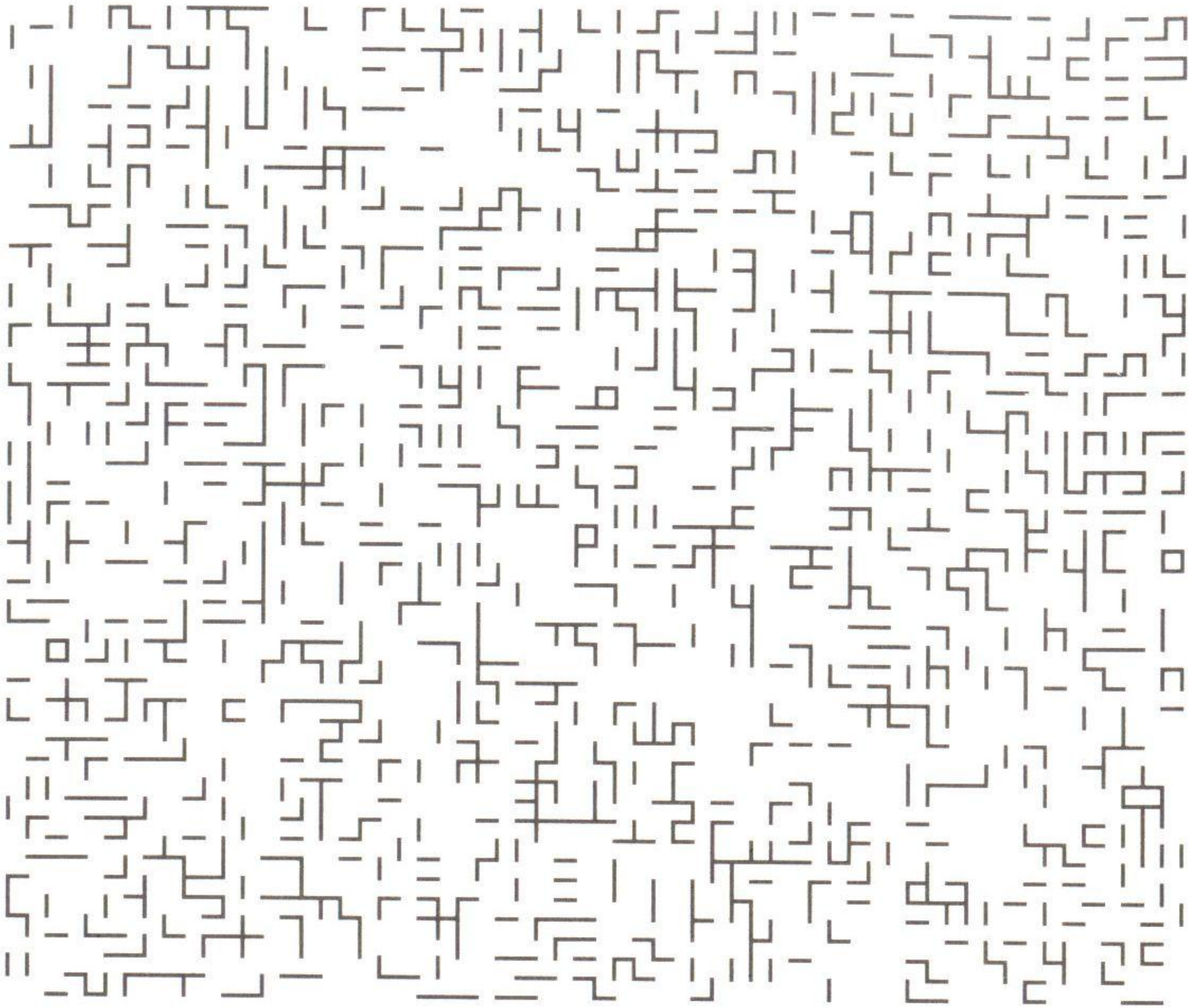
真正有意思的问题: 三维格点Z^3的子图. Terry Lyons

wedge $\{(x, y, z); \quad |z| \le g(x)\}$是常返的
当且仅当 $\sum(x\, g(x))^{-1} = \infty$.

Z^{d+1}的子图$W = \{(x_1, x_2, \ldots\ldots, x_d, z), \; |x_i| \le f_i(z), 1 \le i \le d\}$
是常返的当且仅当

$$\sum_n \left[\prod_{i=1}^{d} f_i(n) \right]^{-1} = \infty.$$

渗流Percolation



(a) $p = 0.25$

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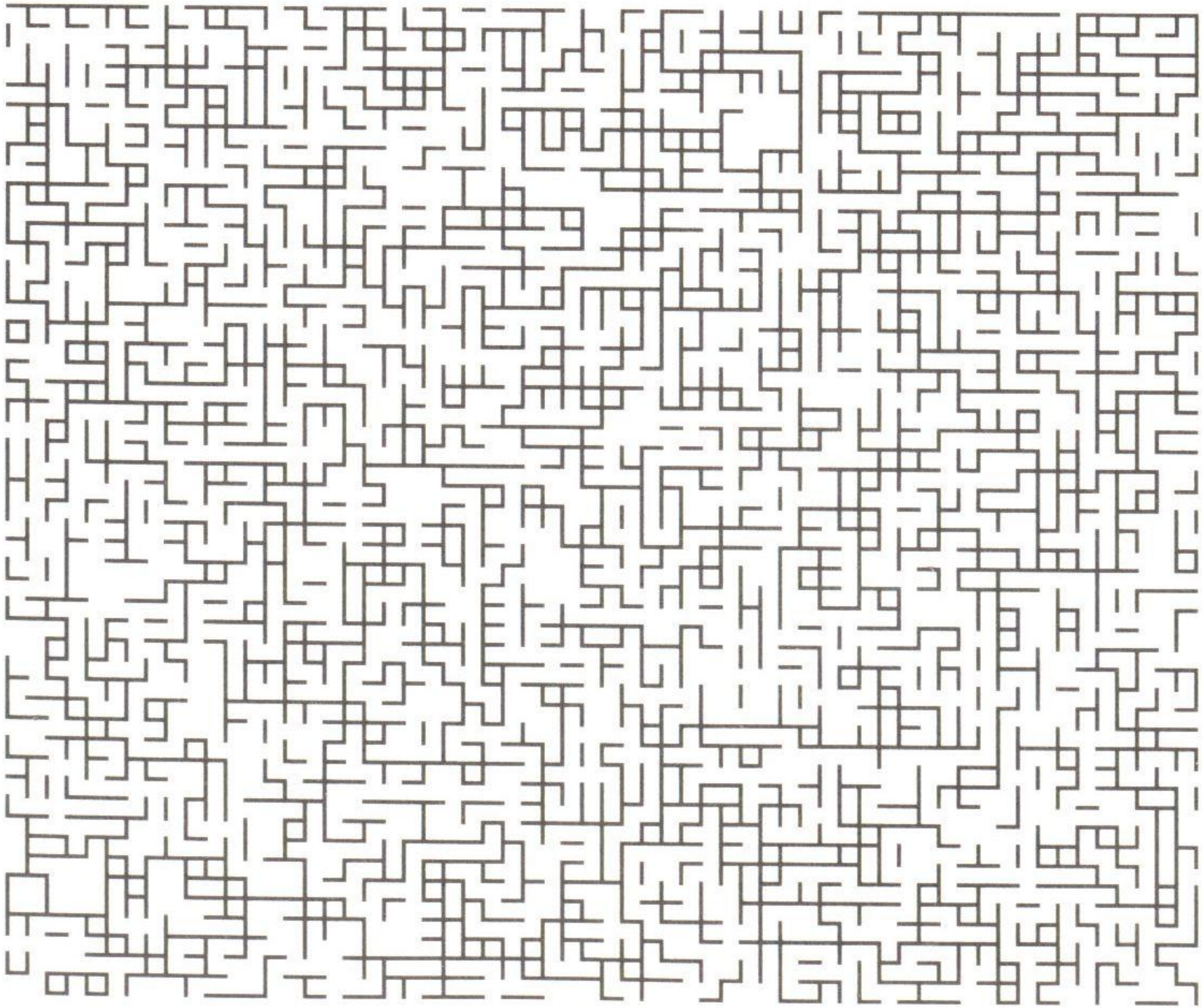
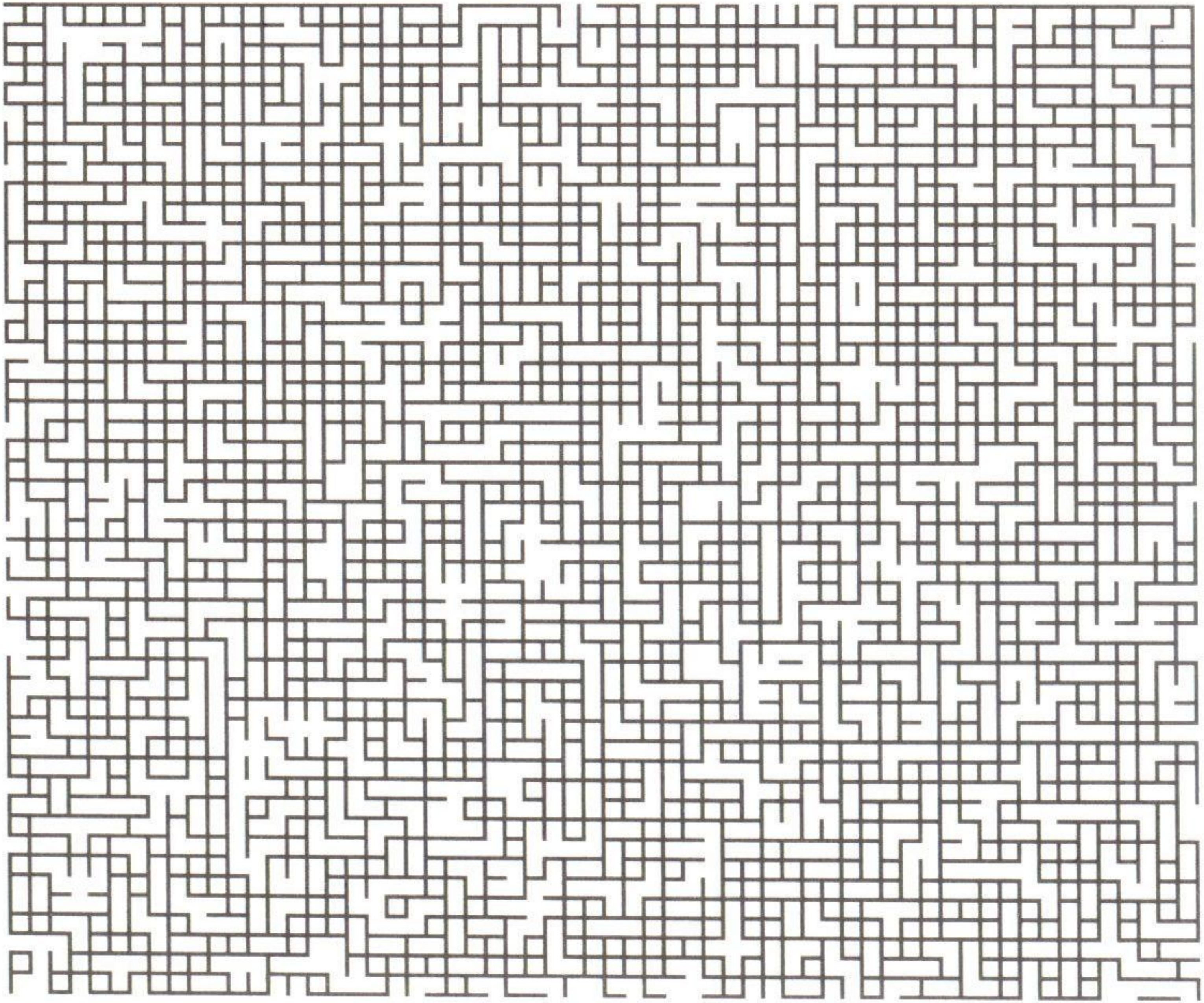


(b) $p = 0.49$



(d) $p = 0.75$

定理 (Grimmett, Kesten & Zhang)
The infinite open cluster of the Bernoulli bond percolation of Z^3 is transient

凭直觉我们相信无穷开簇与原图应当性质相同。

Scherk图是非常返的，
但在临界值附近，无穷开簇是常返的。

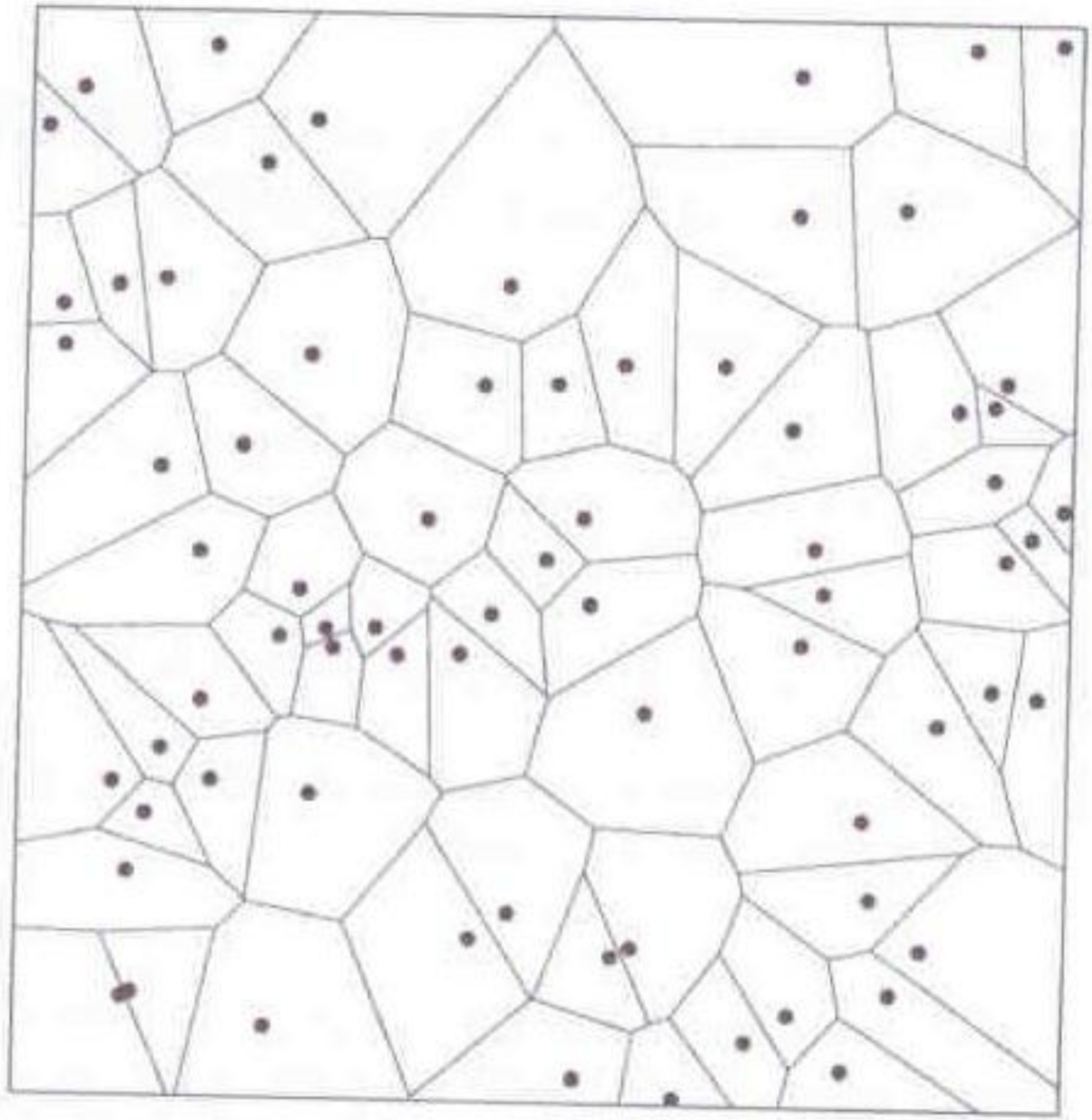
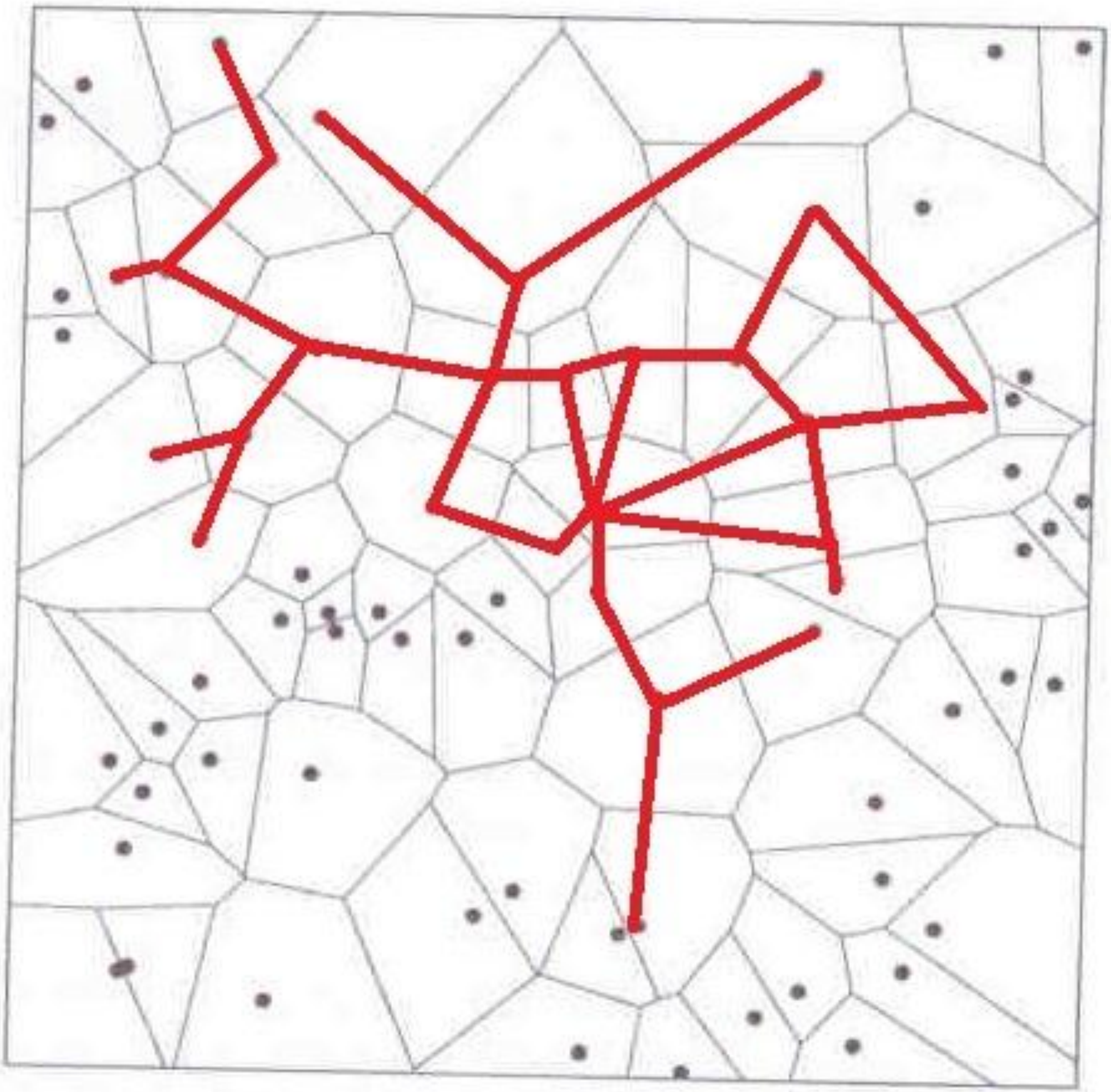


Fig. 1. A sample of Voronoi tessellation. Dotted points represent generating points. Polygon for each point is determined by the perpendicular bisectors between the point and its neighbouring points.



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相遇问题

- 如果图G是齐性的, 对任何x, y ϵ V(G), 可以定义x-y, 则相遇问题可以归结为差图G-G上随机游动的常返问题. Polya.

- 图G上随机游动的相遇问题＝＝> 乘积图G× G上随机游动的常返问题(回到对角线). 难点在于想象不出乘积图G×G.

可以非常返而相遇. 如果p(x,y) = p(y,x)或者零均值, 则

- 1) 非常返 ＝＝> 相遇有限次
- 2) 相遇无限次＝＝>常返
- 3) 常返& 相遇有限次 (Liggett 1974)

- Tom Liggett

出于粒子系统的应用, Liggett(1974)要考虑两个独立的马氏链是否必定相遇, 发现两个常返的马氏链可以永不相遇的第一个例子.

假设S上不可约常返马氏链X的转移概率p(x,y)满足,

$p(x,y) = p(y,x)$

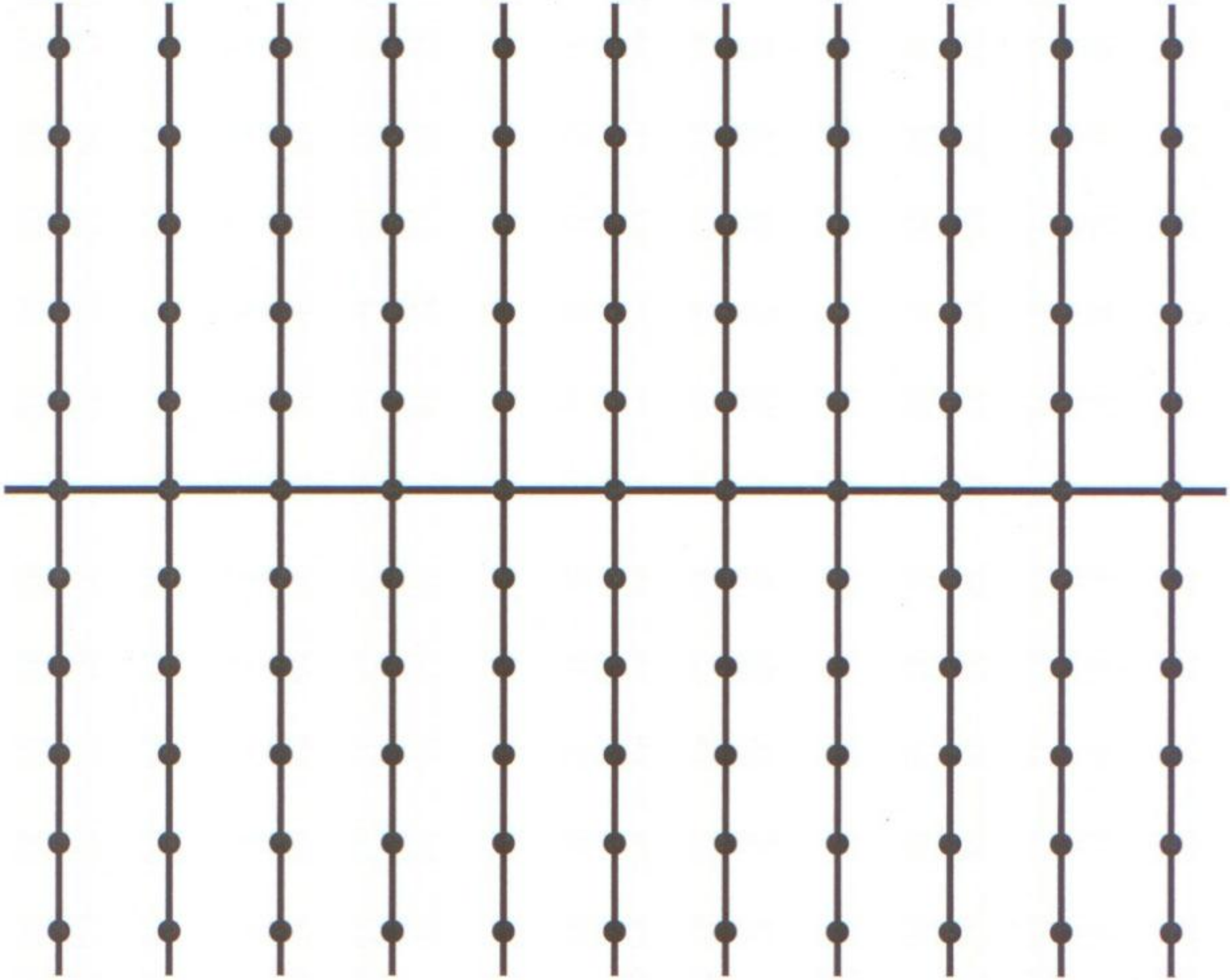
存在$x_0 \in S$, $p(x_0, x_0) > 0$,

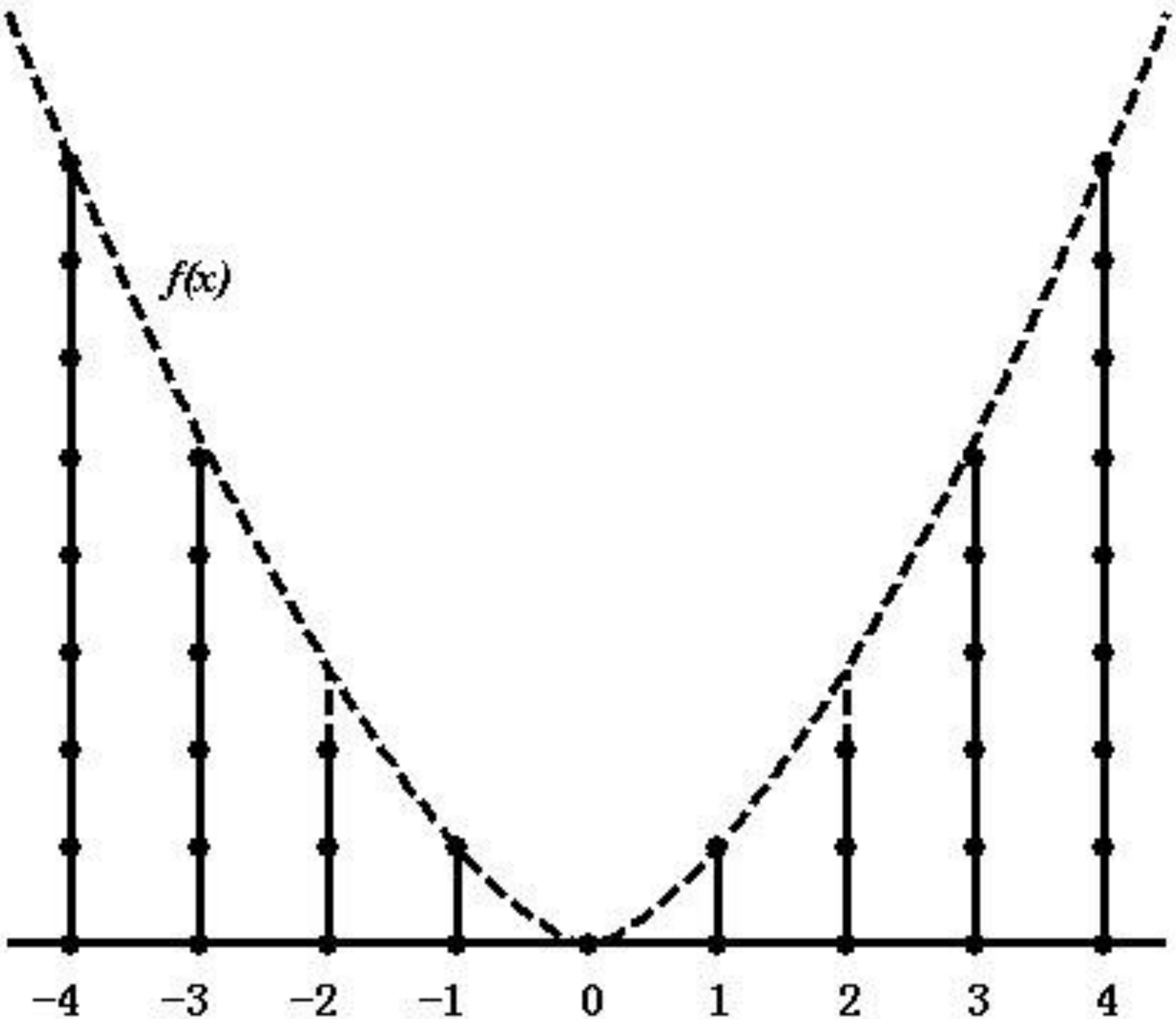
对任何$x \in S$, $x \neq x_0$, $p(x,x) = 0$.

独立耦合的(X,Y)是S×S上非常返的

据此可以构造Z^1上概率分布p_k并定义新的马氏链(x, i), 取值于$S\times Z^1$, 转移概率为.

$$p((x,i),(y,j)) = \begin{cases} p(x,y) & \text{if } i=j,\ x\neq x_0,\ y\neq x_0 \\ p(x_0,x_0)\, p_{j-i} & \text{if } x = y = x_0 \\ 0 & \text{otherwise} \end{cases}$$





f(x)
−4
−3
−2
−1
0
1
2
3
4

1. $f(x) = x^{\alpha}$, $\alpha \le 1/5$, 其上两个随机游动必定相遇. (Chen, Wei & Zhang, 2008).

2. $f(x) = x^{\alpha}$, 若$\alpha < 1$, 则两个随机游动必定相遇;
若$\alpha > 1$, 则两个随机游动不一定相遇. (Barlow, Peres & Sousi, 2010).

3. 若$f(x) \le x \log x$, 则其上两个随机游动必定相遇.
若$f(x) \ge x \log^2 x$, 则两个随机游动不一定相遇.

一般结论 记$g(n) = 1 \vee \max\{f(i), -n \le i \le n\}$, 若

$$\sum_n 1/g(n) = \infty,$$

则两个随机游动必定相遇(Chen & Chen, 2011).

Wedge.

$$W = \{(x_1, x_2, \ldots\ldots, x_d, z),\ |x_i| \le f_i(z),\ 1 \le i \le d\}$$

Terry Lyons证明W上简单随机游动是常返的当且仅当

$$\sum_n \left[\prod_{i=1}^d f_i(n)\right]^{-1} = \infty.$$

陈新兴证明这个条件也是两个随机游动必定相遇的充分必要条件.

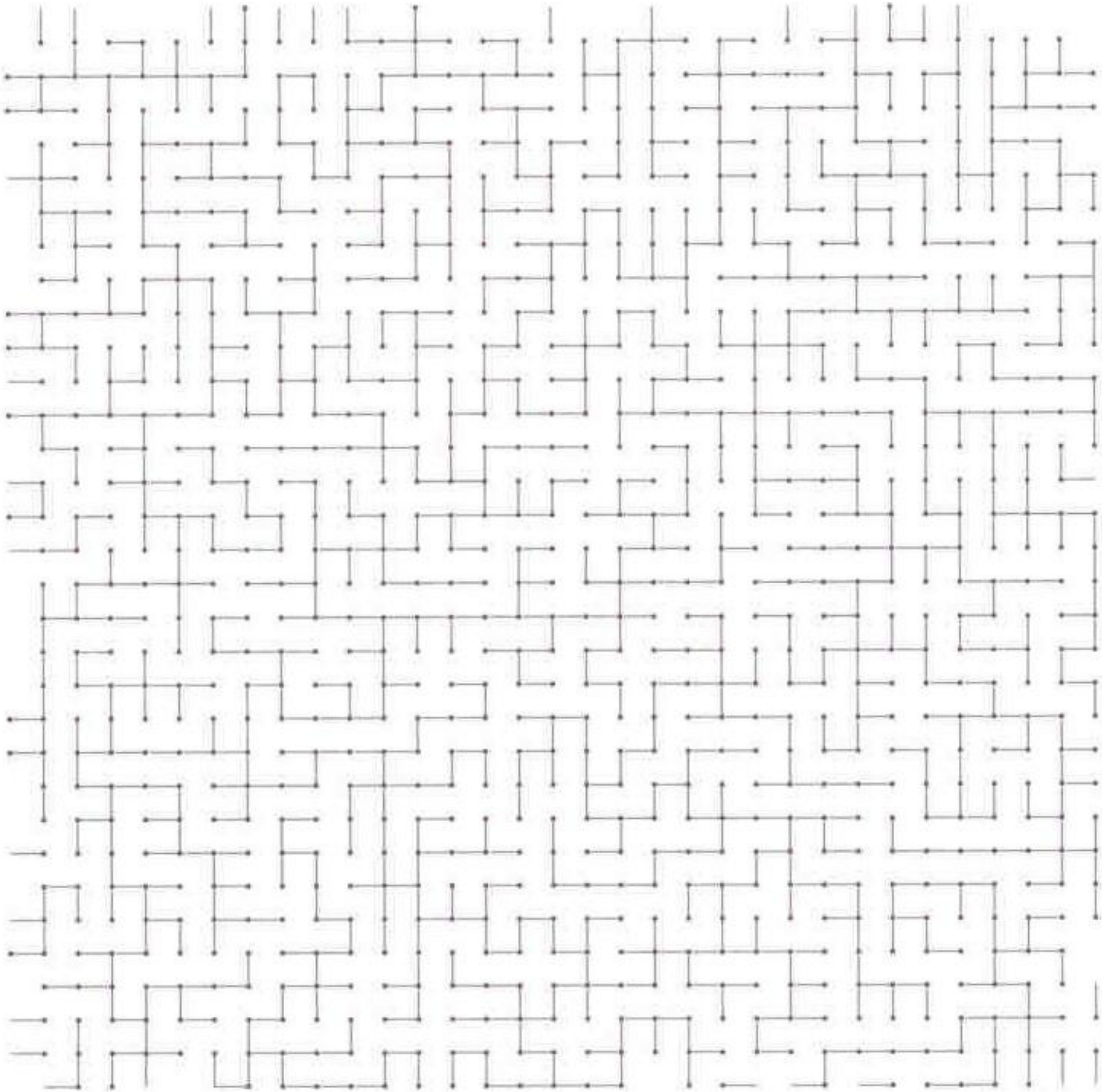
随机环境

1. Z^2上的开簇. 每一条边或以概率p“开”, 以概率1-p "闭", 所有开边组成一些连通子图, 其中最大的一个是无穷的, 考虑其上的简单随机游动. 它是常返的, 其上两个独立随机游动必定相遇.

- 证明基于Barlow的热核估计. (Chen & Chen 2010, Barlow, Peres & Sousi 2010).

2. 随机环境中的随机游动(RWRE), Z^2的每一条边赋以随机权重$\mu_e \geq 1$, $\{\mu_e, e \in E\}$, *i.i.d.* 最近Barlow & Deuschel 得到了这一类随机游动的热核估计.

- 利用此结果可以证明(Shan & Chen, 2012).
- 其上两个独立随机游动必定相遇.
- 随机环境中选举模型的不变分布必是 $a\,\delta_0 + (1-a)\,\delta_1$.



Theorem A. Let $\omega \in \Omega_0$, and $\{X_t\}$, $\{Y_t\}$ be independent variable speed random walks in Z^2 starting from x and y respectively. Then

$$P_\omega (X_t = Y_t \text{ for some } t \geq 1) = 1.$$

Lemma B. Let $\omega \in \Omega_0$, and $\{X_t\}$, $\{Y_t\}$ be independent variable speed random walks in Z^2 starting from x and y respectively. Then

$$P_\omega (X_t = Y_t \text{ for some } t \geq 1) \geq \delta > 0$$

where δ is a constant independent of ω, x and y.

Define the random variable

$$H := \int_{t_0}^{T} \mathbf{1}_{\{X_s = Y_s \in M(\sqrt{s})\}}\, ds.$$

$$E_\omega H \geqq c_9 \log T.$$

$$E_\omega H^2 \leqq (4\pi\, c_3^{\,2} + 2\pi^2\, c_3^{\,4}/c_4) \log^2 T.$$

$$\begin{aligned} P_\omega(H>0) &\geqq (E_\omega H)^2 / E_\omega H^2 \qquad \text{Cauchy-Schwarz} \\ &\geqq (c_9 \log T)^2 / [(4\pi\, c_3^{\,2} + 2\pi^2\, c_3^{\,4}/c_4) \log^2 T] \\ &= (c_9)^2 / [4\pi\, c_3^{\,2} + 2\pi^2\, c_3^{\,4}/c_4]. \end{aligned}$$

定理 (Theorem 1.2 of Barlow & Deuschel, 2010)
Let $d \geqq 2$ and $0 < \sigma < 1$. There exist random variables $\{S_x, x \in Z^d\}$, such that $P(S_x(\omega) \geq n) \leq c_1 \exp(-c_2 n^\sigma)$,
and constants c_i (depending only on d and the distribution of μ_e) such that the following hold.

If $|x-y|^2 \vee t \geq S_x^2$, then
$q_t^\omega(x,y) \leq c_3 t^{-d/2} \exp(-c_4|x-y|^2/t)$ when $t \geq |x-y|$,
$q_t^\omega(x,y) \leq c_3 \exp(-c_4|x-y|(1 \vee \log(|x-y|/t)))$ when $t \leq |x-y|$.

If $t \geq (S_x)^2 \vee |x-y|^{1+\sigma}$, then
$q_t^\omega(x,y) \geq c_5 t^{-d/2} \exp(-c_6|x-y|^2/t)$.

引理　Let $A_n(\omega)$ be the random set defined by

$$A_n(\omega)=\{x: |x| \le n, \quad S_x(\omega) \le 2 \log n \}.$$

Then almost surely there exists a finite random variable $U(\omega)$ such that

$$|A_n(\omega)| \ge c_7 n^2 \quad \text{for any } n \ge U(\omega),$$

where $|A_n(\omega)|$ means the cardinality of set $A_n(\omega)$.

取$T = \exp(2\log t_0/\{1+\sigma\})$，其中$\sigma$是前面定理给出的，并且

$t_0 = [S_x(\omega) \vee S_y(\omega)]^2 + [U(\omega)+ (|x| \vee |y|)(1+12\pi/c_7)]^2.$

从引理B证定理A Let $\delta>0$ be defined as before. Fix $\omega \in \Omega_0$. By Lemma B, there exists a function

$$f: V_2 \times V_2 \to [1, \infty),$$

such that for all $x, y \in V_2$,

$$P_\omega^{(x,y)}(X_t=Y_t \text{ for some } 1<t \le f(x,y)) \ge \delta/2.$$

取$x_0= x$, $y_0=y$ and $t_0=0$.

递归定义 x_i, y_i and t_i, $i \ge 1$ 如次.

假设x_i, y_i and t_i 已经定义.

Let $\boldsymbol{\mathcal{X}}_t$ and $\boldsymbol{\mathcal{Y}}_t$ be two independent continuous-time random walks starting from x_i and y_i. Define

$$X_{i+1}:= \boldsymbol{\mathcal{X}}(f(x_i,y_i)), \qquad y_{i+1}:= \boldsymbol{\mathcal{Y}}(f(x_i,y_i)), \text{ and } t_{i+1}:= t_i+f(x_i,y_i).$$

Define $\mathcal{E}_i$ to be the event that $X_t=Y_t$ for some $t \in (t_i+1, t_{i+1}]$ for $i \ge 0$.

根据强马氏性,

$$P_\omega(\mathcal{E}_i \mid X_t, Y_t, t \le t_i) = P_\omega^{(xi,yi)}(\boldsymbol{\mathcal{X}}_t=\boldsymbol{\mathcal{Y}}_t \text{ for some } 1<t \le f(x_i, y_i)) \ge \delta/2.$$

根据第二Borel-Cantelli引理, $P_\omega(\mathcal{E}_i \text{ infinitely often})=1$.

$$P_\omega(X_t= Y_t \text{ infinitely often}) \ge P_\omega(\mathcal{E}_i \text{ infinitely often})=1.$$

相遇问题的应用

选举模型是状态空间$\{0,1\}^V$上的一个马氏链. 底图G = (V, E), 每个选民只有两种选择，0或1.

选民时时更新自己的选择, 根据周围其他选民的意见.

选举模型的对偶过程是coalescing random walk. 如果p(x,y)是可配称的, 时间倒过来我们可以看见t时刻选民x的意见最初来自何方, 意见的传递路径恰好构成马氏链

如果所有选民意见一致, 全为0或全为1, 就不会再有变化. 换言之, 单点分布δ_0和δ_1是不变分布.

还有其他不变分布吗? 这涉及选民如何参照周围邻居更新自己的选择. 假设位于x的选民以p(x,y)的概率模仿选民y的意见, 若两个独立的马氏链必定相遇, 则所有选民的意见最终必定统一. 即不变分布一定是$a\,\delta_0 + (1-a)\,\delta_1$.

问题

1. Voronoi tessellation

2. 任何 G×F?

3. 单调性

4. 离散时间和连续时间的等价性

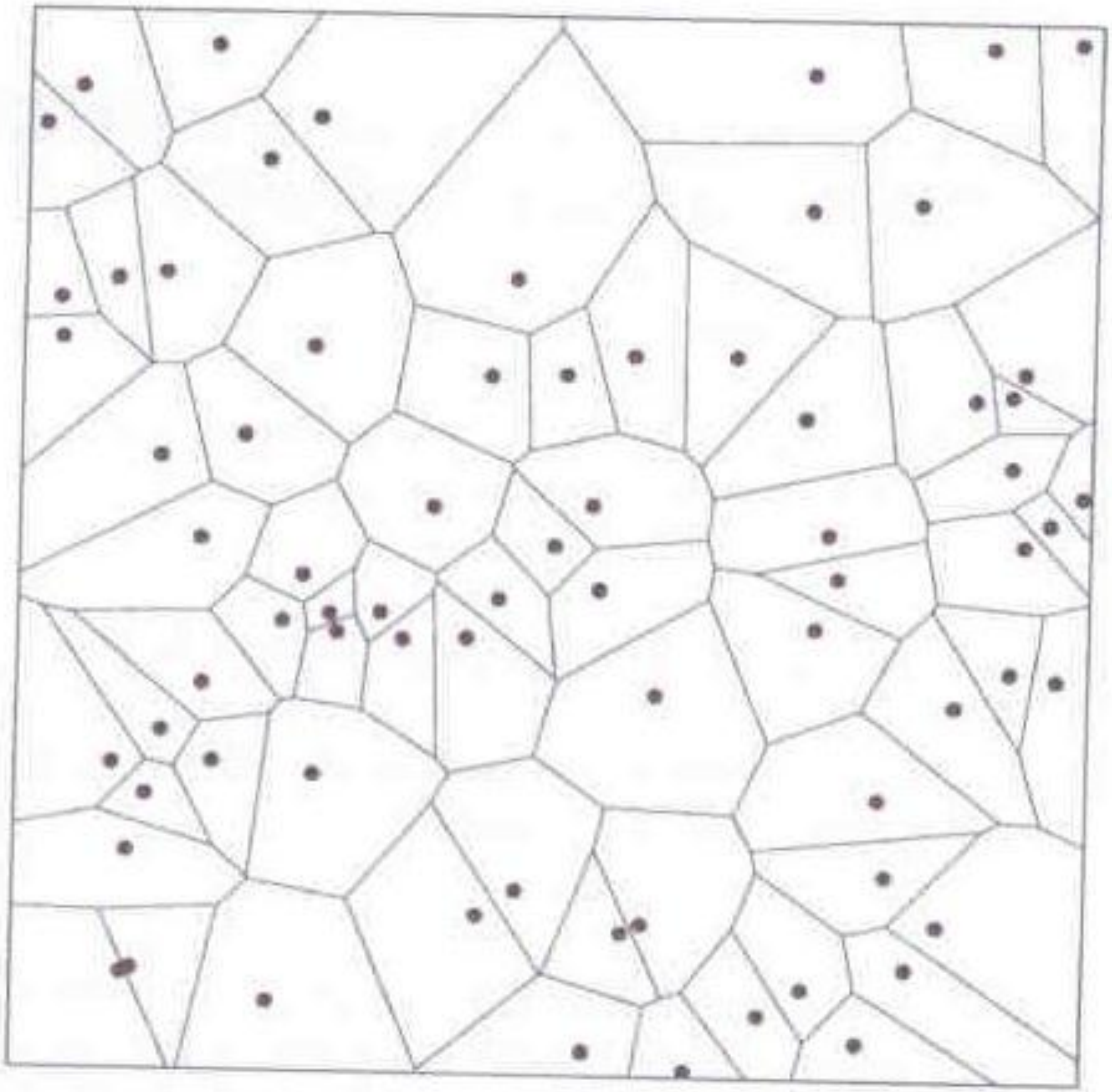
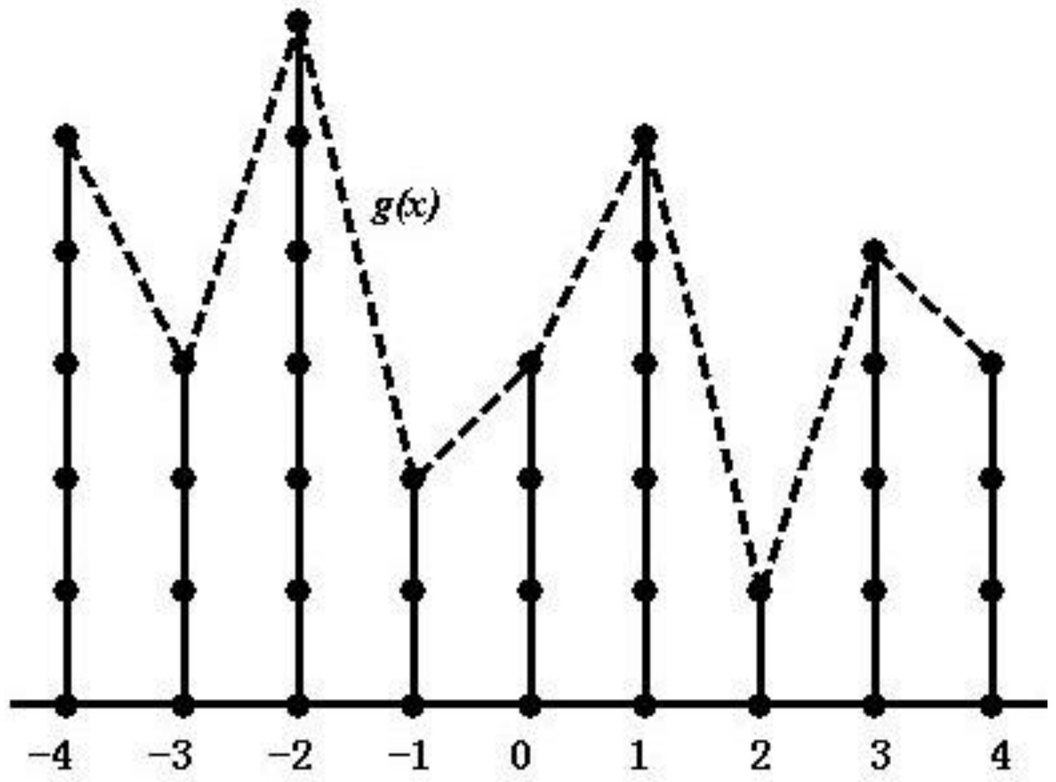
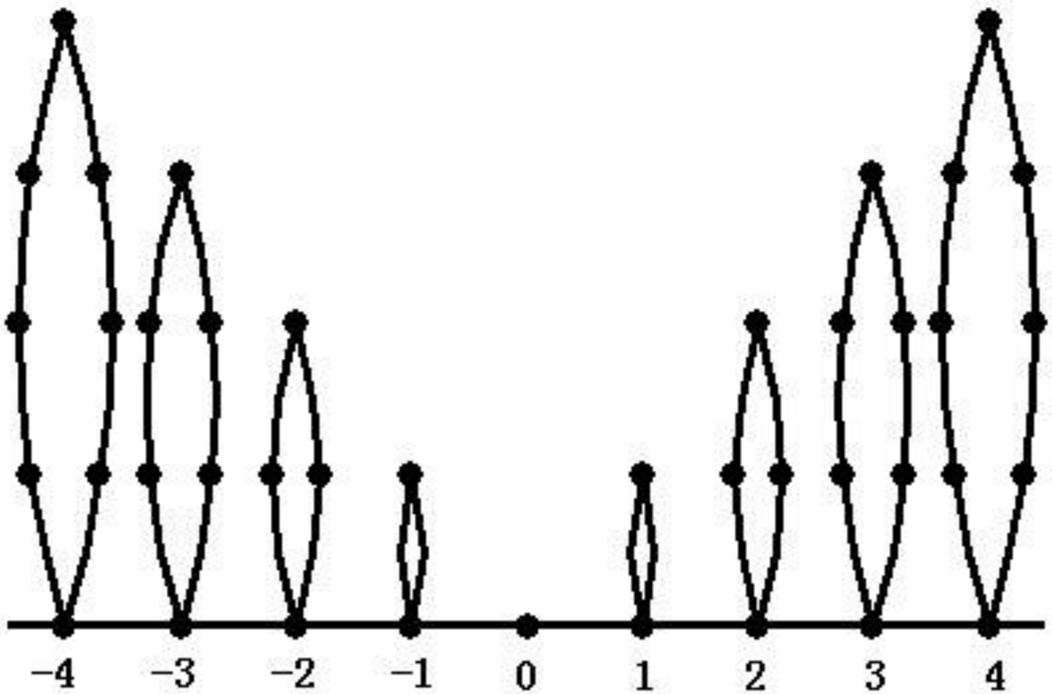
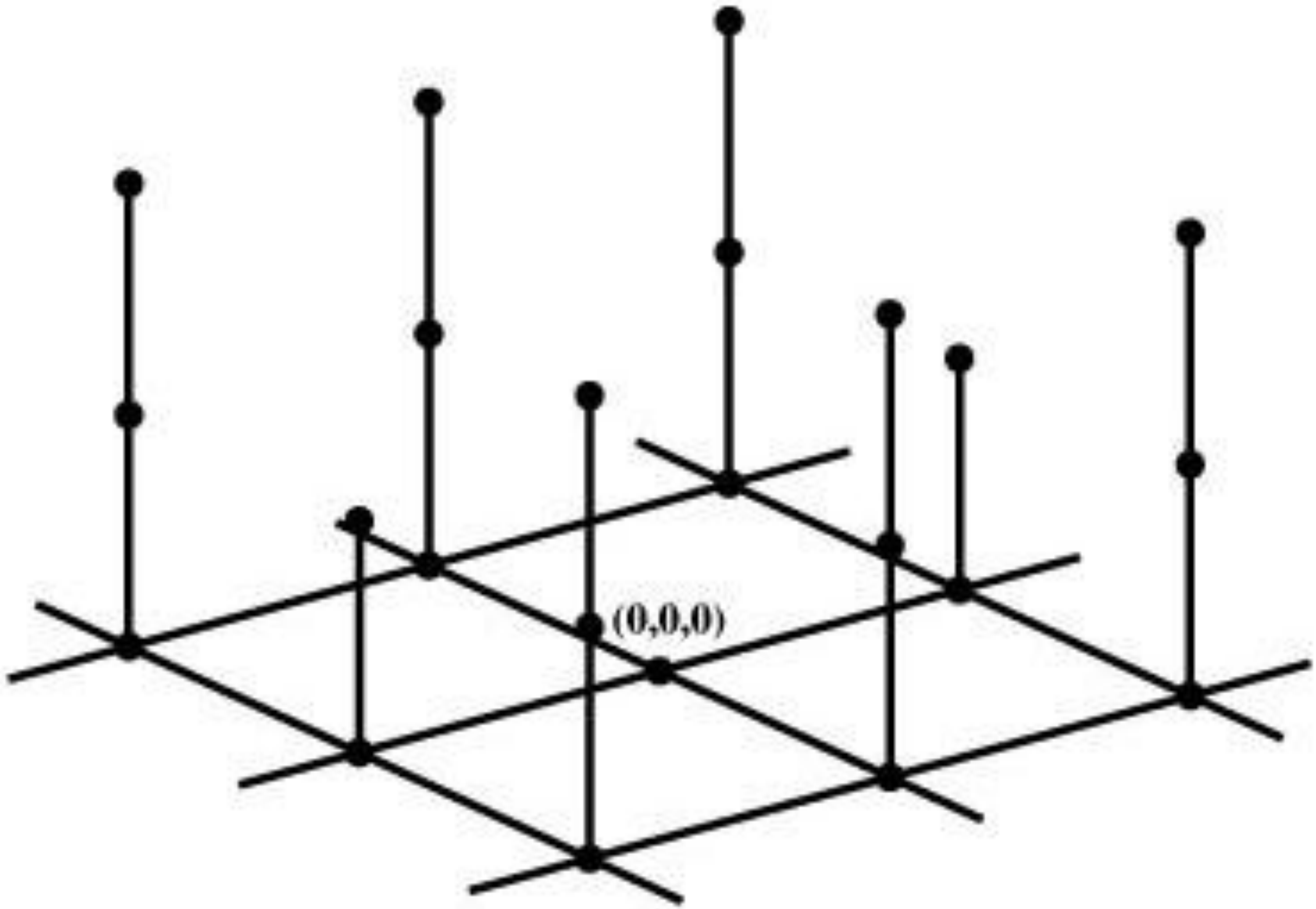


Fig. 1. A sample of Voronoi tessellation. Dotted points represent generating points. Polygon for each point is determined by the perpendicular bisectors between the point and its neighbouring points.


g(x)
−4
−3
−2
−1
0
1
2
3
4


−4
−3
−2
−1
0
1
2
3
4


(0,0,0)

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