The Existence of an Intermediate Phase for the Contact Process on Some Product Graphs

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Abstract In this article the author considers the contact process on the product graph of a homogeneous tree with degree $d \geq 3$ and an arbitrary finite connected simple graph, showing that an intermediate phase for weak survival exists.

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1 Introduction

The contact process on a connected graph $S = (V, E)$ is a continuous-time Markov process ξ_t whose state space is the collection of subsets of V, and with the transition rates

$$\begin{cases} \xi_t \to \xi_t \setminus \{x\} \text{ for } x \in \xi_t \text{ at rate } 1, \\ \xi_t \to \xi_t \cup \{x\} \text{ for } x \notin \xi_t \text{ at rate } \lambda \cdot |\{y \in \xi_t : x \sim y\}|, \end{cases}$$

where $|\cdot|$ denotes the cardinality of a set and $x \sim y$ denotes that there is an edge between vertices x and y. We usually think of ξ_t as the set of sites which are occupied by *infected*(or *active*) particles at time t. Particles die(or become healthy) at rate 1 and are born at a rate equal to the number of neighbors alive multiplied by some fixed parameter λ, with the restriction that no more than one particle may occupy a given site. We shall use ξ_t^A to denote the contact process with starting set A and use ξ_t^x as an abbreviation for $\xi_t^{\{x\}}$, where $x \in V$. Often x will be some distinguished vertex O.

The behavior of the contact process depends crucially on the choice of parameter λ. If

$$\mathbf{P}(\forall t,\ \xi_t^O \neq \emptyset) > 0,$$

we say that the process *survives*; otherwise we say that it *dies out*. If

$$\mathbf{P}(\forall T,\ \exists t > T \text{ with } O \in \xi_t^O) > 0,$$

we say that it *survives strongly*. If the process survives without surviving strongly, we say that it *survives weakly*. Note that neither of these properties depends on the choice of O, since S is connected. By the *monotonicity* of the contact process(see Liggett[1]), we can define two critical values for λ as follows:

$$\begin{cases} \lambda_1 := \inf\{\lambda : \xi_t^O \text{ survives}\}, \\ \lambda_2 := \inf\{\lambda : \xi_t^O \text{ survives strongly}\}. \end{cases}$$

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When we wish to emphasize the graph S, we shall denote these by $\lambda_1(S)$ and $\lambda_2(S)$. It is well known that on an infinite connected graph of bounded degrees, one has $0 < \lambda_1 \leq \lambda_2 < \infty$.

The purpose of this paper is to study the contact process on $\mathbb{T}_d \times G$, where $\mathbb{T}_d$ is a homogeneous tree with degree $d \geq 3$, and G is an arbitrary finite connected simple graph. Here and henceforth we call G a *simple graph* if it contains no loops and no multiple edges. The main result is as follows.

Theorem 1.1 *The contact process on $\mathbb{T}_d \times G(d \geq 3)$ has an intermediate phase in the sense that*

$$\lambda_1 < \lambda_2.$$

Furthermore, we can extend the result in some content. Let H be an isotropic block tree with exponential growth(see Page 1719 of Stacey[2] for the definitions), and let G be an arbitrary finite connected simple graph as above.

Theorem 1.2 *The contact process on $H \times G$ has an intermediate phase in the sense that*

$$\lambda_1 < \lambda_2.$$

The main task of this paper is to prove Theorem 1.1. The idea is enlightened by Stacey[2], who proved the existence of an intermediate phase for the contact process on homogeneous trees and isotropic trees with exponential growth. But the inhomogeneity of G and the existence of cycles both make the proof more difficult. Some new tricks are used in order to deal with the difficulties, especially in Sections 2 and 3 of this paper. We omit the formal proof of Theorem 1.2, since it is similar to the proof of Theorem 1.1. In detail, one can use the trick in Section 2 of Stacey[2] to deal with the inhomogeneity of the isotropic tree and use the trick in this paper to deal with the inhomogeneity of the finite connected graph as well as the existence of cycles in the product graph.

In history, the contact process was first introduced by Harris[3] and has been greatly studied since then. Liggett[1] contains a summary of some important results, as well as numerous references to books and survey papers where further information can be found. An earlier important reference is Liggett[4], which explains why the transition rates given above define a unique process and so forth. The contact process was initially studied on d-dimensional lattices and homogeneous trees. The existence of an intermediate phase is one of the main topics. It has been shown that $\lambda_1(\mathbb{Z}^d) = \lambda_2(\mathbb{Z}^d)$, while $\lambda_1(\mathbb{T}_d) < \lambda_2(\mathbb{T}_d)$ for $d \geq 3$. For $\mathbb{Z}^d$ this follows from results of Bezuidenhout and Grimmett[5]. The result for $\mathbb{T}_d$ was proved by Pemantle[6] for $d \geq 4$ and Liggett[7] for $d = 3$; a simpler proof was given by Stacey[2]. The existence of a phase of weak survival, which does not occur on $\mathbb{Z}^d$, is the principal reason for the interest to study the process on trees.

Next, the contact process has been studied on more general graphs. One reasonable class to consider are the *quasi-transitive* graphs. A graph is said to be *transitive* if the automorphism group acts transitively on the set of vertices(that is, has only one orbit). It is said to be *quasi-transitive* if the action of the automorphism group has only finitely many orbits.

Given an infinite connected graph S, define its *Cheeger's constant* by

$$\iota(S) := \inf\left\{\frac{|\partial H|}{|H|}:\ H \subset S,\ H \text{ is connected},\ 1 \le |H| < \infty\right\},$$

where $\partial H := \{x \in V(S) \setminus V(H):\ \exists y \in V(H),\ s.t.\ x \sim y\}$ is the *boundary* of H. If $\iota(S) = 0$, we say that S is *amenable*; otherwise we say that it's *non-amenable*. Obviously, the lattice $\mathbb{Z}^d$ is amenable for $d \ge 1$, while the homogeneous tree $\mathbb{T}_d$ is non-amenable for $d \ge 3$. Note that the product graphs $\mathbb{T}_d \times G$ in Theorem 1.1 and $H \times G$ in Theorem 1.2 are both infinite quasi-transitive non-amenable simple graphs.

Furthermore, we call a graph S *locally finite* if $\deg(v) < \infty$ for every vertex $v \in V(S)$, where $\deg(v)$ denotes the degree of a vertex $v \in V(S)$.

Pemantle and Stacey[8] conjectured that for the contact process on an infinite locally finite connected quasi-transitive(and hence bounded degrees) graph S, $\lambda_1(S) < \lambda_2(S)$ if and only if S is non-amenable(see Conjecture 5.1 in Pemantle and Stacey[8]). The results of Theorems 1.1 and 1.2 in this paper partially confirm this conjecture and extend the results of Stacey[2] in some content. In fact, when G is a singleton, then Theorems 1.1 and 1.2 in this paper reduce to Theorems 1.3 and 2.0 in Stacey[2] respectively.

The rest of the paper is organized as follows. In Section 2 we will use the subadditive limiting theorem to prove the property of exponential growth and decay for the expected number of infected sites. In Section 3 we will study the process at the first critical value λ_1, showing that it dies out. Finally, we will prove Theorem 1.1 in Section 4.

2 Exponential Growth and Decay

Henceforth, we will denote a vertex in $\mathbb{T}_d \times G$ by (x, y), where $x \in V(\mathbb{T}_d)$ is the $\mathbb{T}_d$-component and $y \in V(G)$ is the G-component. Choose a fixed vertex $O \in V(\mathbb{T}_d)$ as the root of $\mathbb{T}_d$. And denote the vertex set of G by $V(G) = \{z_1, z_2, \cdots, z_m\}$.

Next, we shall prove the property of exponential growth and decay for the expected number of infected sites. The idea comes from Madras and Schinazi[9]. Some new tricks are needed to deal with the inhomogeneity and the existence of cycles.

Proposition 2.1 *There exist constants $c \in \mathbb{R}$ and $C_1, C_2 > 0$ depending on λ, d and m such that, for any $t > 1$, $1 \le k \le m$,*

$$C_1 e^{ct} \le \mathbf{E}(|\xi_t^{(O,z_k)}|) \le C_2 e^{ct}.$$

Furthermore, c is a continuous function of λ.

Proof Without loss of generality, we only prove the case of $k = 1$, the proof of the other cases are the same. Let $\mathcal{F}_t := \sigma\left(\xi_u^{(O,z_1)}:\ u \le t\right)$.

On one hand, for any $s, t > 0$, let $f_s(A) := \mathbf{E}(|\xi_s^A|)$. So if A is a random set, then $f_s(A)$ is a random variable. By additivity of the process(see Liggett[1]) and the Markov property,

$$\mathbf{E}(|\xi_{t+s+1}^{(O,z_1)}| \mid \mathcal{F}_{t+1}) = f_s(\xi_{t+1}^{(O,z_1)}) \le \sum_{(x,y)\in\xi_{t+1}^{(O,z_1)}} \mathbf{E}(|\xi_s^{(x,y)}|). \tag{2.1}$$

Define

$$\alpha := \min_{1\le k\le m} \mathbf{P}(\xi_1^{(O,z_1)} = \{(O,z_k)\}) > 0.$$

Then by the Markov property, for all $(x,y)\in V(\mathbb{T}_d\times G)$,

$$\mathbf{E}(|\xi_{s+1}^{(O,z_1)}|) \ge \alpha\cdot \mathbf{E}(|\xi_s^{(O,y)}|) = \alpha\cdot \mathbf{E}(|\xi_s^{(x,y)}|).$$

Together with (2.1), we get

$$\mathbf{E}(|\xi_{t+s+1}^{(O,z_1)}| \mid \mathcal{F}_{t+1}) \le \sum_{(x,y)\in\xi_{t+1}^{(O,z_1)}} \frac{1}{\alpha}\cdot \mathbf{E}(|\xi_{s+1}^{(O,z_1)}|) = \frac{1}{\alpha}\cdot |\xi_{t+1}^{(O,z_1)}|\cdot \mathbf{E}(|\xi_{s+1}^{(O,z_1)}|).$$

Taking expectation in both sides, we get that for any $s,t>0$,

$$\mathbf{E}(|\xi_{t+s+1}^{(O,z_1)}|) \le \frac{1}{\alpha}\cdot \mathbf{E}(|\xi_{t+1}^{(O,z_1)}|)\cdot \mathbf{E}(|\xi_{s+1}^{(O,z_1)}|).$$

Let $m_t := \frac{1}{\alpha}\cdot \mathbf{E}(|\xi_{t+1}^{(O,z_1)}|)$, then $m_{t+s}\le m_t\cdot m_s$. By the standard subadditive argument(see, for instance, Durrett[10], Page 360-361),

$$c := \lim_{t\to\infty}\frac{1}{t}\log m_t = \inf_{t>0}\frac{1}{t}\log\left[\frac{1}{\alpha}\cdot \mathbf{E}(|\xi_{t+1}^{(O,z_1)}|)\right] = \lim_{t\to\infty}\frac{1}{t}\log[\mathbf{E}(|\xi_{t+1}^{(O,z_1)}|)] \tag{2.2}$$

exists. So

$$\mathbf{E}(|\xi_t^{(O,z_1)}|) \ge \alpha\cdot e^{c(t-1)}$$

for any $t>1$. Let

$$C_1 := \frac{\alpha}{e^c} > 0,$$

then C_1 depends only on λ, d and m. And

$$\mathbf{E}(|\xi_t^{(O,z_1)}|) \ge C_1 e^{ct}$$

for any $t>0$. Furthermore, $\frac{1}{t}\log m_t$ is continuous in λ for any $t>0$. So $c=\inf_{t>0}\frac{1}{t}\log m_t$ is upper-semicontinuous in λ.

On the other hand, given $S\subset V(\mathbb{T}_d\times G)$, define $\pi(S)$ the projection of S on $\mathbb{T}_d$ by

$$\pi(S) := \{v\in V(\mathbb{T}_d):\ \exists j\in V(G) \text{ s.t. } (v,j)\in S\}.$$

For any $(x,y)\in V(\mathbb{T}_d\times G)$ where $x\in V(\mathbb{T}_d)$ and $y\in V(G)$, if we remove all vertices whose $\mathbb{T}_d$-component is x as well as the edges adjacent to them, we are left with d disjoint components. We call each of these components a *branch* adjacent to (x,y). Using an inductive approach, it is not difficult to see that if the projection of the infected set(denoted by A) on $\mathbb{T}_d$ has exactly k elements, then there are more than $k(d-2)$ uninfected disjoint branches that are adjacent to some infected sites. So there must be more that $\frac{k(d-2)}{d}$ different infected sites having at least one uninfected adjacent branch. Classify them according to their G-components and denote the number of them whose G-component is z_i by $n_i(A)(1\le i\le m)$ respectively. Note that if A is a random set, then $n_i(A)(1\le i\le m)$ are random variables. We keep at time $t-1$ only the particles which have at least one uninfected branch. For each of these particles we consider only its offspring located at the same site or on the adjacent branch. By additivity and the

Markov property we get

$$\mathbf{E}(|\xi_{t+s-1}^{(O,z_1)}|) \geq \sum_{i=1}^{m} \mathbf{E}(n_i(\xi_{t-1}^{(O,z_1)})) \cdot b_s^{(i)}, \tag{2.3}$$

where $b_s^{(i)}$ is the expected number of particles of $\xi_s^{(O,z_i)}$ which are located at (O, z_i) or on a given branch adjacent to it. Define

$$\beta := \min_{1 \leq i \leq m} \mathbf{P}(\xi_1^{(O,z_i)} = \{(O, z_1)\}) > 0.$$

Then by symmetry and the Markov property,

$$b_s^{(i)} \geq \frac{1}{d} \cdot \mathbf{E}(|\xi_s^{(O,z_i)}|) \geq \frac{\beta}{d} \cdot \mathbf{E}(|\xi_{s-1}^{(O,z_1)}|) \tag{2.4}$$

for any $1 \leq i \leq m$. Furthermore, since G is a finite graph which has only m vertices, $|\xi_{t-1}^{(O,z_1)}| \leq m \cdot |\pi(\xi_{t-1}^{(O,z_1)})|$ for any $\omega \in \Omega$. Therefore,

$$\mathbf{E}(|\pi(\xi_{t-1}^{(O,z_1)})|) \geq \frac{1}{m} \cdot \mathbf{E}(|\xi_{t-1}^{(O,z_1)}|). \tag{2.5}$$

(2.3), (2.4) and (2.5) together imply

$$\begin{aligned}
\mathbf{E}(|\xi_{t+s-1}^{(O,z_1)}|) &\geq \frac{\beta}{d} \cdot \mathbf{E}(|\xi_{s-1}^{(O,z_1)}|) \cdot \mathbf{E}\left(\sum_{i=1}^{m} n_i(\xi_{t-1}^{(O,z_1)})\right) \\
&\geq \frac{\beta}{d} \cdot \mathbf{E}(|\xi_{s-1}^{(O,z_1)}|) \cdot \frac{d-2}{d} \cdot \mathbf{E}(|\pi(\xi_{t-1}^{(O,z_1)})|) \\
&\geq \frac{\beta(d-2)}{md^2} \cdot \mathbf{E}(|\xi_{t-1}^{(O,z_1)}|) \cdot \mathbf{E}(|\xi_{s-1}^{(O,z_1)}|)
\end{aligned}$$

for any $t, s > 1$. Let $\tilde{m}_t := \dfrac{\beta(d-2)}{md^2} \cdot \mathbf{E}(|\xi_{t-1}^{(O,z_1)}|)$ for $t > 1$. Then $\tilde{m}_{t+s} \geq \tilde{m}_t \cdot \tilde{m}_s$ for any $t, s > 1$. By the standard subadditive argument again,

$$\tilde{c} := \lim_{t \to \infty} \left(-\frac{1}{t} \log \tilde{m}_t\right) = \inf_{t>1} \left[-\frac{1}{t} \log\left(\frac{\beta(d-2)}{md^2} \cdot \mathbf{E}(|\xi_{t-1}^{(O,z_1)}|)\right)\right]$$

exists and equals to $-\lim_{t \to \infty} \dfrac{1}{t} \log[\mathbf{E}(|\xi_{t-1}^{(O,z_1)}|)] = -c$, where c is as defined in (2.2). So

$$\mathbf{E}(|\xi_t^{(O,z_1)}|) \leq \frac{md^2}{\beta(d-2)} \cdot e^{c(t+1)}$$

for any $t > 0$. Let

$$C_2 := \frac{md^2}{\beta(d-2)} \cdot e^c > 0,$$

then C_2 depends only on λ, d and m, and

$$\mathbf{E}(|\xi_t^{(O,z_1)}|) \leq C_2 e^{ct}$$

for any $t > 0$. Furthermore, $\dfrac{1}{t} \log \tilde{m}_t$ is continuous in λ for any $t > 1$. So $c = \sup_{t>1} \dfrac{1}{t} \log \tilde{m}_t$ is lower-semicontinuous in λ.

So we have $C_1 e^{ct} \leq \mathbf{E}(|\xi_t^{(O,z_1)}|) \leq C_2 e^{ct}$ for all $t > 1$, and c is a continuous function of λ, as desired. □

3 The Process at the First Critical Value

The main purpose of this section is to prove the following proposition which is very important to the proof of Theorem 1.1, and use it to explain that the process at the first critical value λ_1 dies out.

Proposition 3.1 *$c = c(\lambda)$ is as defined in (2.2), then $c(\lambda_1) = 0$ and when $\lambda = \lambda_1$,*

$$C_1 \le \mathbf{E}(|\xi_t^{(O,z_k)}|) \le C_2$$

for any $t > 1$ and $1 \le k \le m$, where C_1 and C_2 are two positive constants depending only on d and m.

Again, without loss of generality, we suppose $k = 1$, the proof of the other cases are the same. The idea of the proof of Proposition 3.1 is enlightened by Morrow, Schinazi and Zhang[11]. Extra tricks are used in order to deal with the inhomogeneity and the existence of cycles here. Some lemmas are needed before the proof.

We begin by recalling the *graphical construction* of the contact process. Readers can consult Liggett[1] for more details. We associate each site $x \in V(\mathbb{T}_d \times G)$(for simplicity we do not use the componential form here) with $\deg(x)+1$ independent Poisson processes, one with rate 1 and the $\deg(x)$ others with rate λ, where $\deg(x)$ denotes the degree of x. Make these Poisson processes independent from site to site. For each $x \in V(\mathbb{T}_d \times G)$, let $\{T_n^{x,k} :\ n \ge 1\}$, $k = 0, 1, 2, \cdots, \deg(x)$ be the arrival times of these $\deg(x)+1$ processes respectively. The process $\{T_n^{x,0}\}$ has rate 1, the others have rate λ. For each $x \in V(\mathbb{T}_d \times G)$ and $n \ge 1$ we write a δ mark at the point $(x, T_n^{x,0})$ while we draw arrows from $(x, T_n^{x,k})$ to $(x_k, T_n^{x,k})$ if $k \ge 1$, where $\{x_k :\ k = 1, 2, \cdots, \deg(x)\}$ are the neighbors of x. We say that there is a path from (x, s) to (y, t) if there is a sequence of times $s_0 = s < s_1 < \cdots < s_{m+1} = t$ and spatial locations $x_0 = x, x_1, \cdots, x_m = y$ so that for $i = 1, 2, \cdots, m$, there is an arrow from x_{i-1} to x_i at time s_i and the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ do not contain any δ. Use the notation $\{(x, s) \to (y, t)\}$ to denote the event that there is a path from (x, s) to (y, t). To construct the contact process from the initial configuration A where there is one particle at each site of A, we let $\xi_t^A(y) = 1$ if there is a path from $(x, 0)$ to (y, t) for some x in A, which is also denoted by $\{(A, 0) \to (y, t)\}$.

Next, we will give some notation. We still use the componential form to denote a site henceforth. Consider the homogeneous tree $\mathbb{T}_d$, denote by $\mathcal{B}_\mathbb{T}$ the connected component of the subgraph obtained by removing a distinguished subset of $d-1$ edges, each having an endpoint at the root O. Then denote $\mathcal{B} := \mathcal{B}_\mathbb{T} \times G$. We construct the *severed contact process in* $\mathcal{B}$ by considering only the paths(in the graphical construction) in $\mathcal{B}$. Denote by $\{((x_1, y_1), s) \xrightarrow{\mathcal{B}} ((x_2, y_2), t)\}$ the event that there is a path from $((x_1, y_1), s)$ to $((x_2, y_2), t)$ inside $\mathcal{B}$. Denote by

$$\{(A, s) \xrightarrow{\mathcal{B}} ((x_2, y_2), t)\} := \bigcup_{(x_1,y_1)\in A} \{((x_1, y_1), s) \xrightarrow{\mathcal{B}} ((x_2, y_2), t)\},$$

where $A \subset V(\mathbb{T}_d \times G)$. Then denote by $\eta_t^{(x,y)}$ the severed contact process in $\mathcal{B}$, starting with one particle at the site $(x, y) \in V(\mathcal{B})$.

Using the graphical construction described above, we can construct the two processes $\xi_t^{(O,z_i)}$ and $\eta_t^{(O,z_i)}$ simultaneously for any $1 \le i \le m$, but for the severed contact process we only use the arrows that are located in $\mathcal{B}$. For $i = 1, 2, \cdots, m$, define $\{\rho_t^{(i)},\ t \geq 0\}$ to be the projection process of $\eta_t^{(O,z_i)}$ on $\mathcal{B}_{\mathbb{T}}$. That is to say, for any $x \in V(\mathcal{B}_{\mathbb{T}})$, $\rho_t^{(i)}(x) = 1$ if and only if there exists $y \in V(G)$ such that $\eta_t^{(O,z_i)}(x, y) = 1$.

Lemma 3.1 *If $c(\lambda) > 0$, then for any constant $K > 0$, there exists $T = T(K, \lambda) > 1$ such that*

$$\mathbf{E}(|\rho_T^{(i)}|) \geq K \tag{3.1}$$

for any $i = 1, 2, \cdots, m$.

Proof Take $(x, y) \in V(\mathcal{B})$ where $x \neq O$. Let O' be the neighbor of O in $\mathcal{B}_{\mathbb{T}}$. For each $k = 1, 2, \cdots, m$, denote the kth Poisson process to be the Poisson process from (O, z_k) to (O', z_k) with parameter λ. By monotonicity of the contact process(see Liggett[1]) and the Markov property,

$$\begin{aligned}
&\mathbf{P}(\xi_t^{(O,z_1)}(x,y) = 1)\\
\leq\ & \int_0^t \mathbf{P}\left[\bigcup_{k=1}^m (\{\text{the } k\text{th Poisson process has a jump in } ds\} \cap \{((O', z_k), s) \xrightarrow{\mathcal{B}} ((x,y),t)\})\right]\\
\leq\ & \int_0^t \mathbf{P}\left[\left(\bigcup_{k=1}^m \{\text{the } k\text{th Poisson process has a jump in } ds\}\right) \cap \{(O' \times G, s) \xrightarrow{\mathcal{B}} ((x,y,t)\}\right]\\
=\ & \int_0^t \mathbf{P}\left(\bigcup_{k=1}^m \{\text{the } k\text{th Poisson process has a jump in } ds\}\right) \cdot \mathbf{P}[(O' \times G, s) \xrightarrow{\mathcal{B}} ((x,y),t)]\\
\leq\ & \int_0^t m\lambda ds \cdot \mathbf{P}[(O' \times G, s) \xrightarrow{\mathcal{B}} ((x,y),t)]\\
=\ & m\lambda \cdot \int_0^t \mathbf{P}[(O' \times G, s) \xrightarrow{\mathcal{B}} ((x,y),t)]ds.
\end{aligned}$$

We use $O' \times G$ as an abbreviation for $\{O'\} \times G$ now and henceforth. Sum over all $(x, y) \in V(\mathcal{B})$ where $x \neq O$ to get

$$\sum_{(x,y)\in V(\mathcal{B}),\ x\neq O} \mathbf{P}(\xi_t^{(O,z_1)}(x,y) = 1) \leq m\lambda \cdot \int_0^t \sum_{(x,y)\in V(\mathcal{B}),\ x\neq O} \mathbf{P}[(O' \times G, s) \xrightarrow{\mathcal{B}} ((x,y),t)]ds.$$

So we have

$$\sum_{(x,y)\in V(\mathcal{B})} \mathbf{P}(\xi_t^{(O,z_1)}(x,y) = 1) \leq m\lambda \cdot \int_0^t \sum_{(x,y)\in V(\mathcal{B})} \mathbf{P}[(O' \times G, s) \xrightarrow{\mathcal{B}} ((x,y),t)]ds + m. \tag{3.2}$$

By symmetry and Proposition 2.1,

$$\sum_{(x,y)\in V(\mathcal{B})} \mathbf{P}(\xi_t^{(O,z_1)}(x,y) = 1) = \mathbf{E}(|\xi_t^{(O,z_1)} \cap \mathcal{B}|) \geq \frac{1}{d} \cdot \mathbf{E}(|\xi_t^{(O,z_1)}|) \geq \frac{C_1}{d} \cdot e^{c(\lambda)t}.$$

Also note that

$$\sum_{(x,y)\in V(\mathcal{B})} \mathbf{P}[(O' \times G, s) \xrightarrow{\mathcal{B}} ((x,y),t)] = \mathbf{E}(|\eta_{t-s}^{O' \times G}|).$$

So by (3.2),

$$\frac{1}{m\lambda}\left[\frac{C_1}{d} e^{c(\lambda)t} - m\right] \leq \int_0^t \mathbf{E}(|\eta_{t-s}^{O' \times G}|)ds = \int_0^t \mathbf{E}(|\eta_s^{O' \times G}|)ds \leq t \cdot \sup_{s\leq t} \mathbf{E}(|\eta_s^{O' \times G}|).$$

In other words,

$$\sup_{s\le t}\mathbf{E}(|\eta_s^{O'\times G}|)\ge\frac{1}{m\lambda t}\left[\frac{C_1}{d}\cdot e^{c(\lambda)t}-m\right].\tag{3.3}$$

The right-hand side of (3.3) tends to infinity as $t\to\infty$ since $c(\lambda)>0$. So for any constant $K>0$, we can find a time $T>1$ depending on K and λ such that

$$\mathbf{E}(|\eta_{T-1}^{O'\times G}|)\ge\frac{m}{\gamma}\cdot K,$$

where $\gamma:=\min_{1\le i\le m}\mathbf{P}(\eta_1^{(O,z_i)}=O'\times G)>0$. Then by the Markov property,

$$\mathbf{E}(|\eta_T^{(O,z_i)}|\ge\gamma\cdot\mathbf{E}(|\eta_{T-1}^{O'\times G}|)\ge mK\qquad(\forall 1\le i\le m).\tag{3.4}$$

Note that for all $\omega\in\Omega$, $|\eta_t^{(O,z_i)}|\le m\cdot|\rho_t^{(i)}|$ for all $t>0$. So by (3.4),

$$\mathbf{E}(|\rho_T^{(i)}|)\ge\frac{1}{m}\cdot\mathbf{E}(|\eta_T^{(O,z_i)}|)\ge K$$

for any $1\le i\le m$, as desired. □

Next we will show that (3.1) is enough to prove that the severed contact process survives with positive probability. The idea is to construct the multi-type branching processes which lie below the contact process. The new processes are easier to analyze than the contact process itself.

By the argument in the proof of Proposition 2.1, for any finite subset $S\subset V(\mathbb{T}_d\times G)$, there are at least $\frac{d-2}{d}|\pi(S)|-1$ branches which satisfy the following four properties:

(a) are adjacent to some point in S,

(b) have no point in S,

(c) are contained in $\mathcal{B}$, and

(d) are disjoint from each other.

For simplicity we denote $B(S)$ the set of vertices in S which emanate the branches described above(satisfying properties (a) to (d)). So $|B(S)|\ge\frac{d-2}{d}|\pi(S)|-1$. Fix $T>1$ which will be specified later. Using the graphical construction, for any $1\le i\le m$ we define a new process $\tilde{\eta}_t^{(i)}$ as follows. $\tilde{\eta}_t^{(i)}$ evolves like $\eta_t^{(O,z_i)}$ up to time T. At time T we make all the particles of $\tilde{\eta}_T^{(i)}$ which are not in $B(\tilde{\eta}_T^{(i)})$ become healthy, and restrict the spatial evolution of the remaining particles in the following way. Each particle in $B(\tilde{\eta}_T^{(i)})$ generates a process for which births are allowed only on the empty branches described above(satisfying (a) to (d)). At time T, we create at least $\frac{d-2}{d}|\pi(\tilde{\eta}_T^{(i)})|-1$ severed contact processes which are independent of one another. Repeat the preceding step at all times kT and only keep the particles in $B(\tilde{\eta}_{kT}^{(i)})$. Between times kT and $(k+1)T$, the process evolves according to the graphical construction. If we define the discrete-time process $Z_k^{(i)}:=|\tilde{\eta}_{kT}^{(i)}|$ $(k\ge1,\ 1\le i\le m)$ and $Z_0^{(i)}:=1$ $(1\le i\le m)$, then $\{Z_k^{(i)},\ k\ge0\}$ is a multi-type branching process. It can be described as follows. There are $m\ge1$ different types of individuals. Each type has its *offspring scheme* which decides the distribution of its offsprings' number and types. At generation 0 there is only one individual of type $i(1\le i\le m)$. Then it gives birth to some offsprings according to the offspring scheme of type i and repeat it generation by generation. The process evolves in the way that the individuals of the same type have the same offspring distribution, and all different individuals give birth to offsprings independently. $Z_k^{(i)}$ denotes the total number of individuals in generation k. Note that when $m = 1$, $\{Z_k^{(i)}\}$ is just the ordinary Galton-Watson branching process.

Furthermore, for any $1 \le i \le m$,

$$\mu_i := \mathbf{E}(Z_1^{(i)}) = \mathbf{E}(|\tilde{\eta}_T^{(i)}|) \ge \frac{d-2}{d} \cdot \mathbf{E}(|\rho_T^{(i)}|) - 1. \tag{3.5}$$

The next lemma gives a sufficient condition for the survival of the multi-type branching process.

Lemma 3.2 *If $\mu_i > 1$ for every $1 \le i \le m$, where μ_i is as defined in (3.5), then*

$$\mathbf{P}(Z_k^{(i)} \ge 1, \ \forall k \ge 0) > 0$$

for any $1 \le i \le m$.

Remark 3.1 *One can use Theorem 2 on Page 186 of Athreya and Ney[12] to prove Lemma 3.2 after checking the conditions of that theorem. However, in order to avoid the tedious checking procedure which based on matrix theory, we present a direct proof here, which probably seems simpler.*

Proof of Lemma 3.2 For $1 \le i \le m$, let $p_{l_1,\cdots,l_m}^{(i)}$ be the probability that the individual of type i has l_j offsprings of type j $(1 \le j \le m)$. For $i = 1, 2, \cdots, m$, define

$$\Phi_i(t) := \sum_{l_1 \ge 0, \ \cdots, \ l_m \ge 0} p_{l_1, \ \cdots, \ l_m}^{(i)} \cdot t^{l_1 + \cdots + l_m}$$

for $0 \le t \le 1$. Then for any $1 \le i \le m$,

$$\Phi_i(0) \ge 0, \ \Phi_i(1) = 1.$$

Furthermore, using differentiation one can get that Φ_i is continuous, strictly increasing and convex on the interval $[0, 1]$. Then define

$$\Phi(t) := \max_{1 \le i \le m} \Phi_i(t)$$

for $0 \le t \le 1$. It is easy to see that Φ is continuous and strictly increasing on $[0, 1]$ with

$$\Phi(0) \ge 0, \ \Phi(1) = 1.$$

Since $\Phi_i^{'}(1) = \mu_i > 1$ for any $1 \le i \le m$, there exists $\delta_i > 0$ such that $\Phi_i(t) < t$ for any $t \in [1 - \delta_i, 1)$. Take

$$\delta := \min_{1 \le i \le m} \delta_i > 0,$$

then $\Phi(t) < t$ for any $t \in [1 - \delta, 1)$. Together with the fact that $\Phi(0) \ge 0$, we get that there exists $\rho \in [0, 1 - \delta)$ such that $\Phi(\rho) = \rho$. For $1 \le i \le m, \ k \ge 0$, define

$$\tau_k^{(i)} := \mathbf{P}(Z_k^{(i)} = 0),$$

then

$$\tau_k^{(i)} \nearrow \tau_\infty^{(i)} = \mathbf{P}(\exists k, \ Z_k^{(i)} = 0)$$

as $k \to \infty$. Furthermore, define

$$\tau_k := \max_{1\le i\le m} \tau_k^{(i)}$$

for $1 \le k \le \infty$. Then $\tau_k \nearrow \tau_\infty$ as k tends to infinity. Note that for any $1 \le i \le m$,

$$\tau_{k+1}^{(i)} = \sum_{l_1\ge 0,\ \cdots,\ l_m\ge 0} p^{(i)}_{l_1,\ \cdots,\ l_m} \cdot (\tau_k^{(1)})^{l_1} \cdot \cdots \cdot (\tau_k^{(m)})^{l_m} \le \Phi_i(\tau_k) \le \Phi(\tau_k).$$

Then

$$\tau_{k+1} = \max_{1\le i\le m} \tau_{k+1}^{(i)} \le \Phi(\tau_k)$$

for any $k \ge 0$. Since $\tau_0 = 0 \le \rho$, then

$$\tau_1 \le \Phi(\tau_0) \le \Phi(\rho) = \rho.$$

Using induction to get $\tau_k \le \rho$ for all k, so

$$\tau_\infty = \lim_{k\to\infty} \tau_k \le \rho < 1.$$

Furthermore,

$$\mathbf{P}(\exists k,\ Z_k^{(i)} = 0) = \tau_\infty^{(i)} < 1$$

for any $1 \le i \le m$. In other words,

$$\mathbf{P}(\forall k,\ Z_k^{(i)} > 0) > 0$$

for any $1 \le i \le m$, as desired. □

The next lemma we will need is a well-known fact about Markov chains with absorbing states. Readers can see (2.5) on Page 46 of Liggett[1] for the proof in the $\mathbb{Z}^d$ case. The proof of it in our case is almost the same and is therefore omitted here. Define

$$\Omega_\infty := \{|\xi_t^{(O,z_1)}| \ge 1,\ \forall t > 0\}. \tag{3.6}$$

Lemma 3.3 *$\lim_{t\to\infty} |\xi_t^{(O,z_1)}| = \infty$ a.s. on Ω_∞.*

Proof of Proposition 3.1 On one hand, when $\lambda > \lambda_1$, the process $\xi_t^{(O,z_1)}$ survives. So $\mathbf{P}(\Omega_\infty) > 0$, where Ω_∞ is as defined in (3.6). By Lemma 3.3, $\lim_{t\to\infty} |\xi_t^{(O,z_1)}| = \infty$ almost surely on Ω_∞. So, by Fatou's lemma,

$$\liminf_{t\to\infty} \mathbf{E}(|\xi_t^{(O,z_1)}|) \ge \mathbf{E}(\liminf_{t\to\infty} |\xi_t^{(O,z_1)}|) = +\infty.$$

Then $c(\lambda) \ge 0$. Otherwise, by Proposition 2.1, $\mathbf{E}(|\xi_t^{(O,z_1)}|) \le C_2 e^{c(\lambda)t} \to 0$ as $t \to \infty$, a contradiction. By the continuity of $c(\lambda)$ in λ, we get $c(\lambda_1) \ge 0$.

On the other hand, fix λ such that $c(\lambda) > 0$ and take constant $K > \dfrac{2d}{d-2}$. Then by Lemma 3.1, there exists $T = T(K, \lambda) > 1$ such that

$$\mathbf{E}(|\rho_T^{(i)}|) \ge K > \frac{2d}{d-2}.$$

Together with (3.5) to get

$$\mu_i \ge \frac{d-2}{d} \cdot \mathbf{E}(|\rho_T^{(i)}|) - 1 > 1$$

for any $1 \le i \le m$. Then by Lemma 3.2, $\{Z_k^{(1)}\}$ survives with positive probability, so does $\eta_t^{(O,z_1)}$ and therefore $\xi_t^{(O,z_1)}$. Then

$$\mathbf{P}(|\xi_t^{(O,z_1)}| > 0,\ \forall t \ge 0) > 0.$$

Therefore, $\lambda \ge \lambda_1$.

We have shown that if $c(\lambda) > 0$ then $\lambda \ge \lambda_1$. In other words, if $\lambda < \lambda_1$ then $c(\lambda) \le 0$. By the continuity of $c(\lambda)$ in λ, we get $c(\lambda_1) \le 0$. So $c(\lambda_1) = 0$. Furthermore, by Proposition 2.1, when $\lambda = \lambda_1$,

$$C_1 \le \mathbf{E}(|\xi_t^{(O,z_1)}|) \le C_2$$

for any $t > 1$, where C_1 and C_2 are two positive constants depending only on d and m, as desired. □

Corollary 3.1 *If $\lambda = \lambda_1$, then $\xi_t^{O\times G}$ dies out.*

We use $O \times G$ as an abbreviation for $\{O\} \times G$ now and henceforth. The proof of this corollary is omitted since the fact that the survival property does not depend on the initial state if it is finite, and the proof of the singleton case is quite similar to the first paragraph in the proof of Proposition 3.1.

4 The Existence of an Intermediate Phase

In this section we will prove our main result, Theorem 1.1. Our approach is similar to the one used by Stacey[2].

We will give some new definitions first. It will greatly simplify some calculations if the homogeneous tree $\mathbb{T}_d$ is arranged so that every vertex has one neighbor above it and $d-1$ neighbors below it. We can then assign a *level* to each vertex in such a way that the root O has level 0 and any vertex in level l has one neighbor in level $l-1$ and $d-1$ neighbors in level $l+1$. For $n \in \mathbb{Z}$, we shall use $\mathcal{L}_n$ to denote the set of all vertices in level n. Of course, each set $\mathcal{L}_n$ is infinite. Use $l(x)$ to denote the level of a vertex $x \in V(\mathbb{T}_d)$.

Having arranged the vertices in levels, we now define the *weight* of a vertex $(x, y) \in V(\mathbb{T}_d \times G)$ by

$$w_\alpha(x, y) := \alpha^{l(x)}, \tag{4.1}$$

where $\alpha > 0$ is to be specified later; we shall often use $w(x, y)$ as the abbreviation for $w_\alpha(x, y)$. The weight of a set of vertices is defined to be the sum of the weights of all vertices in the set. This arrangement of the tree and assignment of weights appears in Liggett[7].

Having made these definitions, it is easy to establish the following result, whose proof is a slight extension of the proof of Proposition 1.0 in Stacey[2] and is therefore omitted here.

Proposition 4.1 *Let $\{\xi_t^{O\times G}\}$ be the contact process on $\mathbb{T}_d \times G$ with parameter λ and starting set $O \times G$, where $\mathbb{T}_d$ $(d \ge 3)$ is a homogeneous tree and G is a finite connected graph. Suppose that for some $t_0 > 0$ and some weight function w_α,*

$$\mathbf{E}(w_\alpha(\xi_{t_0}^{O\times G})) = \beta < 1.$$

Then

$$\mathbf{P}(\omega \in \Omega: \ \exists T = T(\omega), \ s.t. \ \forall t \geq T, \ (O \times G) \cap \xi_t^{O\times G} = \emptyset) = 1,$$

so a fortiori, $\lambda \leq \lambda_2$.

We shall also need one technical result about the behavior of the weight function.

Lemma 4.1 *Let* $\{\xi_t^{O\times G,\ \lambda}: \ t \geq 0\}$ *be the contact process on* $\mathbb{T}_d \times G$ *with parameter* λ *and starting set* $O \times G$*, where* $\mathbb{T}_d$ $(d \geq 3)$ *is a homogeneous tree and* G *is a finite connected graph. Let* $w = w_\alpha$ *be a weight function as above and let* T *be some fixed time. Then the function*

$$\lambda \to \mathbf{E}(w(\xi_T^{O\times G,\ \lambda}))$$

is continuous.

The result of this lemma is rather obvious since the function $\mathbf{E}(w(\xi_T^{O\times G,\ \lambda}))$ depends only on the process for finite time periods. One can refer to Liggett[1] Page 39-40 for details.

Proof of Theorem 1.1 Let $\xi_t^{O\times G}$ be the contact process at the first critical value λ_1 with starting set $O \times G$. Let $w(\cdot)$ be the weight function as defined by (4.1) with $\alpha = \dfrac{1}{\sqrt{d-1}}$. Let

$$\mathcal{D}_{n,k} := \{(x,y) \in V(\mathbb{T}_d \times G): \ |x - O|_{\mathbb{T}} = n, \ y = z_k\}$$

for $n \geq 0, \ k = 1, 2, \cdots, m$, where $|\cdot|_{\mathbb{T}}$ denotes the graphic norm on $\mathbb{T}_d$, that is, the shortest length of paths joining the two points in $\mathbb{T}_d$. Then

$$|\mathcal{D}_{n,k}| = d(d-1)^n.$$

Furthermore,

$$\mathbf{E}(w(\xi_t^{O\times G})) = \mathbf{E}\left(\sum_{(x,y)\in\xi_t^{O\times G}} w(x,y)\right) = \mathbf{E}\left(\sum_{n\geq 0}\sum_{k=1}^{m}\sum_{(x,y)\in\mathcal{D}_{n,k}} w(x,y)\cdot \mathbf{1}_{\{(x,y)\in\xi_t^{O\times G}\}}\right)$$
$$= \sum_{n\geq 0}\sum_{k=1}^{m}\sum_{(x,y)\in\mathcal{D}_{n,k}} w(x,y)\cdot \mathbf{P}((x,y)\in\xi_t^{O\times G}).$$

Let

$$a_{n,k} := \frac{w(\mathcal{D}_{n,k})}{|\mathcal{D}_{n,k}|}$$

for $n \geq 0, \ k = 1, 2, \cdots, m$. Note that by symmetry, $\mathbf{P}((x,y) \in \xi_t^{O\times G})$ are the same for any $(x,y) \in \mathcal{D}_{n,k}$, denote it by $p_{n,k}$. So

$$\mathbf{E}(w(\xi_t^{O\times G})) = \sum_{n\geq 0}\sum_{k=1}^{m} p_{n,k}\cdot w(\mathcal{D}_{n,k}) = \sum_{n\geq 0}\sum_{k=1}^{m} a_{n,k}\cdot|\mathcal{D}_{n,k}|\cdot p_{n,k}$$
$$= \sum_{n\geq 0}\sum_{k=1}^{m} a_{n,k}\cdot \sum_{(x,y)\in\mathcal{D}_{n,k}} \mathbf{P}((x,y)\in\xi_t^{O\times G})$$
$$= \sum_{n\geq 0}\sum_{k=1}^{m} a_{n,k}\cdot \mathbf{E}(|\xi_t^{O\times G}\cap\mathcal{D}_{n,k}|). \tag{4.2}$$

We can calculate the value of each $a_{n,k}$ $(n \geq 0,\ k = 1, 2, \cdots, m)$ accurately. First classify the vertices in $\mathcal{D}_{n,k}$ according to their generation number as follows(note that $\mathcal{D}_{n,k} \subset \bigcup_{i=-n}^{n} \mathcal{L}_i$ for $k = 1, 2, \cdots, m$). For any $k = 1, 2, \cdots, m$,

$$|\mathcal{D}_{n,k} \cap \mathcal{L}_n| = (d-1)^n,$$

$$|\mathcal{D}_{n,k} \cap \mathcal{L}_{n-2}| = (d-2)(d-1)^{n-2},$$

$$|\mathcal{D}_{n,k} \cap \mathcal{L}_{n-4}| = (d-2)(d-1)^{n-3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$|\mathcal{D}_{n,k} \cap \mathcal{L}_{n-(2i-2)}| = (d-2)(d-1)^{n-i},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$|\mathcal{D}_{n,k} \cap \mathcal{L}_{-n+4}| = (d-2)(d-1)^{1},$$

$$|\mathcal{D}_{n,k} \cap \mathcal{L}_{-n+2}| = (d-2)(d-1)^{0},$$

$$|\mathcal{D}_{n,k} \cap \mathcal{L}_{-n}| = (d-1)^{0} = 1.$$

Note that the pattern varies slightly at the start and finish.

So for any $n \geq 0,\ k = 1, 2, \cdots, m$, using our choice of $\alpha = \dfrac{1}{\sqrt{d-1}}$ to get

$$\begin{aligned} w(\mathcal{D}_{n,k}) &= \alpha^{-n} + (d-1)^n \alpha^n + \sum_{i=2}^{n} (d-2)(d-1)^{n-i} \alpha^{n-(2i-2)} \\ &= (d-1)^{\frac{n}{2}} + (d-1)^{\frac{n}{2}} + \sum_{i=2}^{n} (d-2)(d-1)^{\frac{n-2}{2}} \\ &= 2(d-1)^{\frac{n}{2}} + (n-1)(d-2)(d-1)^{\frac{n-2}{2}}. \end{aligned}$$

Recall that $|\mathcal{D}_{n,k}| = d(d-1)^{n-1}$. So

$$a_{n,k} = \frac{w(\mathcal{D}_{n,k})}{|\mathcal{D}_{n,k}|} = \frac{2(d-1) + (n-1)(d-2)}{d(d-1)^{\frac{n}{2}}} \to 0$$

as $n \to \infty$ since $d \geq 3$. Therefore, we can take N large enough, such that when $n \geq N$, $a_{n,k} \leq \dfrac{\varepsilon}{mC_2}$ for any $k = 1, 2, \cdots, m$, where C_2 is the positive constant depending only on d and m which appears in Proposition 3.1. Then we split up the right-hand side of (4.2) as

$$\sum_{0 \leq n < N} \sum_{k=1}^{m} \sum_{(x,y) \in \mathcal{D}_{n,k}} w(x,y) \cdot \mathbf{P}((x,y) \in \xi_t^{O \times G}) + \sum_{n \geq N} \sum_{k=1}^{m} a_{n,k} \cdot \mathbf{E}(|\xi_t^{O \times G} \cap \mathcal{D}_{n,k}|). \qquad (4.3)$$

The second term in (4.3) is easy to bound. By Proposition 3.1,

$$\begin{aligned} \sum_{n \geq N} \sum_{k=1}^{m} a_{n,k} \cdot \mathbf{E}(|\xi_t^{O \times G} \cap \mathcal{D}_{n,k}|) &\leq \frac{\varepsilon}{mC_2} \cdot \sum_{n \geq N} \sum_{k=1}^{m} \mathbf{E}(|\xi_t^{O \times G} \cap \mathcal{D}_{n,k}|) \\ &\leq \frac{\varepsilon}{mC_2} \cdot \mathbf{E}(|\xi_t^{O \times G}|) \leq \frac{\varepsilon}{mC_2} \cdot \sum_{k=1}^{m} \mathbf{E}(|\xi_t^{(O,z_k)}|) \leq \varepsilon \end{aligned}$$

for any $t > 1$. Next we shall bound the first term of (4.3). By Corollary 3.1, the contact process with parameter $\lambda = \lambda_1$ dies out. So for any $(x, y) \in \bigcup_{0\le n<N} \bigcup_{k=1}^{m} \mathcal{D}_{n,k}$, $\mathbf{P}((x,y) \in \xi_t^{O\times G}) \to 0$ as $t \to \infty$. Since $\bigcup_{0\le n<N} \bigcup_{k=1}^{m} \mathcal{D}_{n,k}$ has only finitely many vertices, the first term in (4.3) is at most ε for sufficiently large t. Therefore, for some value t_0,

$$\mathbf{E}(w(\xi_{t_0}^{O\times G})) \le 2\varepsilon.$$

Then by Lemma 4.1, there exists $\lambda^* > \lambda_1$ such that

$$\mathbf{E}(w(\xi_{t_0}^{O\times G,\ \lambda^*})) \le 3\varepsilon.$$

Since ε was arbitrary, we can choose it in such a way that $3\varepsilon < 1$. At this time,

$$\mathbf{E}(w(\xi_{t_0}^{O\times G,\ \lambda^*})) < 1.$$

By Proposition 4.1, $\lambda^* \le \lambda_2$. So we get $\lambda_1 < \lambda_2$, which completes the proof. □

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