Beijing Seminar–September 15, 2006

Pointwise Fourier inversion: a classical topic revisited

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Background. One dimensional Fourier analysis of piecewise smooth functions features the following properties:

Convergence: The F.S. converges to the average of the left and right limits.

Gibbs-Wilbraham phenomenon: In the neighborhood of a jump, the partial sums overshoot the jump by approx 9% of the jump.

Localization: If the function is zero on an interval, then the F.S. converges to zero there.

In higher dimensions these properties are no longer valid. We illustrate in the following model cases:

I. 2D Fourier-Bessel series: $f(x) = \sum_{n \geq 1} A_n J_0(z_n x)$ where $J_0(z_n) = 0$

II. 3D Fourier-Bessel series: $f(x) = \sum_{n \geq 1} B_n \frac{\sin n\pi x}{n\pi x}$

In both cases we have the radial eigenfunctions of the Laplace operator in the respective dimensions, where we have Dirichlet boundary conditions on the the unit sphere. When we dispense with boundary conditions, the spectral analysis of the 3D Laplace operator is provided by

III. Fourier integral in 3D: $f(x) = \int_{R^3} \widehat{f}(\mu) e^{i<\mu,x>} \, d\mu$

I. 2D F-Bessel series of

$$f(x) = 1 \text{ for } 0 \le x \le 1$$

Note the usual Gibbs phenomenon at $x = 1$ and the strange behavior at $x = 0$. This is due to the *slow convergence* at $x = 0$ (speed $n^{-1/2}$) in contrast to speed n^{-1} at $x \ne 0$.

II. 3D F-Bessel series of

$$f(x) = 1 \text{ for } 0 \leq x \leq 1$$

In this case the partial sums satisfy

$$\liminf_{M \to \infty} F_M(0) = 0, \limsup_{M \to \infty} f_M(0) = 2$$

We still have the Gibbs overshoot at $x = 0$, but now the series *diverges* at $x = 0$ where the function is smooth.

III. 3D Fourier integral of

$f(x) = 1$ for $0 \leq |x| \leq 1$ and $f(x) = 0$ otherwise

I.1 Abs. inversion of F. T. on R^n

$$f \in L^1(R^n), \quad \hat{f}(\xi) := \int_{R^n} f(x) e^{-2\pi i x \cdot \xi} \, dx$$

Gaussian (heat kernel) example:

$$H_t(x) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}}, \quad \hat{H}_t(\xi) = e^{-4\pi^2 t |\xi|^2}$$

This basic example allows one to write the heat kernel convolution of a general $f \in L^1(R^n)$ as a Fourier integral:

$$\int_{R^n} H_t(y) f(x-y) \, dy = \int_{R^n} \hat{H}_t(\xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi.$$

$\forall f \in L^1(R^n)$ the l.h.s. has a $t \downarrow 0$ limit a.e. $x \in R^n$. If also $\hat{f} \in L^1(R^n)$ then the r.h.s. tends to a continuous limit when $t \downarrow 0$ and we have the absolutely convergent a.e. formula

$$f(x) = \int_{R^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \qquad (*)$$

By re-defining f on a null set, then $(*)$ holds everywhere. For example, if f has $n+1$ derivatives in $L^1(R^n)$, then $\hat{f}(\xi) = O(|\xi|^{-(n+1)})$. Can assume less with Bernstein techniques.

I.2 Bochner's approach

Bochner (1931) studied the pointwise convergence of the spherical partial sums

$$S_M f(x) := \int_{|\xi| \leq M} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi. \qquad (I.2.1)$$

Spherical Fourier inversion is the statement

$S_M f(x) \to f(x)$ when $M \to \infty$. To study this, introduce the *spherical mean value*

$$\bar{f}_x(r) = \frac{1}{\omega_{n-1}} \int_{|\omega|=1} f(x + r\omega) dS_\omega$$

where the integration is with respect to the induced surface measure on the unit sphere and ω_{n-1} is the total surface measure of the unit sphere.

By transforming the integrals, we can write

$$S_M f(x) = \omega_{n-1} \int_0^\infty D_M^n(r) \bar{f}_x(r) r^{n-1} \, dr \qquad (**)$$

Spherical Dirichlet kernel is

$$D_M^n(z) := \int_{|\xi| \leq M} e^{-2\pi i x \cdot \xi} \, d\xi = \frac{M^n J_{n/2}(2\pi M |z|)}{(M|z|)^{n/2}}$$

and $J_{n/2}$ is a standard Bessel function.

The principal properties of the Dirichlet kernel are the following, where C_n denotes a generic constant:

$$
\begin{aligned}
(i) D_M^n(0) &= C_n M^n \\
(ii) r > 0 \Rightarrow D_M^n(r) &= C_n M^{\frac{n-1}{2}} [\cos(2\pi M r - \theta_n) + O(\tfrac{}{}) \\
(iii) |D_M^n(r)| &\leq \frac{C_n M^n}{(1 + Mr)^{\frac{n+1}{2}}} \\
(iv) D_M^n(r) &= \frac{-1}{2\pi r} \frac{d}{dr} D_M^{n-2}(r) \quad n \geq 3, r > 0
\end{aligned}
$$

Theorem I.2.1(Bochner) Let $n = 2k + 1$ and let $r \to \bar{f}_x(r)$ be absolutely continuous, and its derivatives of orders through $k - 1$ and

$\int_0^\infty r^{j-1} |\bar{f}_x^{(j)}(r)| \, dr < \infty$ for $1 \le j \le k$. Then

$$\lim_{M \to \infty} S_M f(x) = \bar{f}_x(0+0) \qquad (I.2.5)$$

Hence pointwise Fourier inversion requires only $(n-1)/2$ derivatives, instead of $n+1$. The proof uses integration by parts applied to (**).

Example Let $n = 3$ and $f(x) = 1_{[0,a]}(|x|)$, the indicator function of the ball of radius a centered at $0 \in R^3$. Then $r \to \bar{f}_x(r)$ is Lipschitz continuous when $x \ne 0$, but has a unit jump at $r = a$ when $x = 0$. Theorem I.2.1 predicts convergence at $x \ne 0$ and *suggests* difficulties at $x = 0$. In fact, a direct computation shows that

$$S_M f(0) = -\frac{2}{\pi} \sin(2\pi M a) + 2 \int_0^a \frac{\sin 2\pi M r}{\pi r} \, dr$$

which oscillates boundedly when $M \to \infty$.

Kahane (1995) put this example into a general context, as follows:

K_1: Suppose that $K \subset R^3$ is a bounded region with analytic boundary. If there exists $x \in K$ so that

$$\lim_M S_M 1_K(x) \qquad \text{fails to exist,}$$

then K is a sphere centered at x

K_2: Given a finite set $P_1, \ldots, P_k \in R^3$, there exists a bounded region with C^∞ boundary such that $\lim_M S_M 1_K(x)$ fails to exist whenever $x \in \{P_1, \ldots, P_k\}$

I.3 Piecewise smooth viewpoint

Definition: $f \in L^1(R^n)$ is piecewise smooth with respect to $x \in R^n$ if there is a finite set $0 = a_0 < a_1 < \cdots < a_K$ such that $r \to \bar{f}_x(r)$ is absolutely continuous on each subinterval, with right and left limits and the endpoints, together with its derivatives of orders $\leq [\frac{n+1}{2}]$; furthermore $\int_0^\infty r^{j-1} |\bar{f}_x^{(j)}(r)| \, dr < \infty$ for $1 \leq j \leq [\frac{n+1}{2}]$.

Definition: The **differentiability index** of $f \in L^1(R^n)$ with respect to $x \in R^n$, is defined as follows:

If $r \to \bar{f}_x(r)$ is discontinuous, $J(f,x) = -1$. Otherwise

$$J(f;x) = \max\{j \geq 0 : r \to \bar{f}_x(r) \in C^{(j)}\}$$

Example For 1_B in R^3, $J(1_B; 0) = -1$ whereas $J(1_B; x) = 0$ for $x \neq 0$.

Theorem 3.1 [P, 1994] Suppose that $f \in L^1(R^n)$ is piecewise smooth with respect to $x \in R^n$. Then $\lim_M S_M f(x)$ exists if and only if $J(f; x) \geq [\frac{(n-3)}{2}]$, in which case the limit $= \bar{f}_x(0+0)$. Otherwise we have divergence:

$$S_M f(x) - \bar{f}_x(0+0) \sim M^\nu \text{ where } \nu = (n-5)/2 - J(f; x) \geq$$

This theorem is quite general, giving a dimension-dependent relation between pointwise Fourier inversion and smoothness of the spherical mean value.

I.4 Fourier series of radial functions

We can attempt to mimick the above steps on the n-torus, defining

$$\hat{f}(k) \;=\; \int_{T^n} e^{-2\pi i x \cdot k} f(x)\, dx$$

$$\tilde{S}_M f(x) \;=\; \sum_{|k| \le M} e^{2\pi i k \cdot x} \hat{f}(k) = \int_{T^n} \tilde{D}_M(x - y) f(y)\, dy$$

However $\tilde{D}_M(x) := \sum_{|k| \le M} e^{2\pi i k \cdot x} \ne \tilde{D}(|x|)$. Therefore we cannot reduce to a one-dimensional problem by integration-by-parts. To obtain some partial results, we restrict to radial functions $f(x) = F(|x|)$ where $F \in L^1(0, \pi)$ or more generally let f be the periodization of a radial function:

$f(x) = \sum_{k \in Z^n} F(|x - k|)$, so that the Fourier coefficients of f are also radial and we can apply the Poisson sumation formula.

Theorem I.4.1 (P,S,T, 1993) Suppose that $n = 3$ and that $F \in C^2[0, a]$ for some $a > 0$. Then the spherical partial sum $\tilde{S}_M f(0)$ converges when $M \to \infty$ if and only if $F(a) = 0$.

Convergence at $x \neq 0$ was obtained by Kuratsubo(1998).

In higher dimensions we have the following explicit example:

Example If $n > 3$ and $F = 1_{[0,a]}(|x|)$, then

$$\tilde{S}_M f(0) = C_n M^{\frac{n-3}{2}} (\cos(2\pi M a) + O(1/M)) \qquad M \to \infty$$

Kuratsubo (1999) found some positive results in higher dimensions. Let $\phi : [0, a] \to R$ be a function of bounded variation, set equal to zero for $t > a$ and $F_\phi(x) = \sum_{m \in Z^n} \phi(|x + m|)$ be the associated periodized function.

Theorem K1. If $1 \leq n \leq 4$ and $x \neq 0$, then

$$\lim_M S_M f(x) = \sum_{|x+m|<a} \frac{\phi(|x+m|^+) + \phi(|x-m|^-)}{2}$$

$$+ \frac{\phi(a^+) + \phi(a^-)}{2} \sum_{|x+m|=a} 1 \qquad (K1)$$

Theorem K2. If $n = 5$ and $x \notin Q^n$ (non-rational coordinates) then $K1$ holds.

Theorem K3. If $n \geq 6$ then (K1) holds for almost all $x \in T^n$.

I.5 Results on spheres and hyperbolic spaces

Fourier analysis on (M, g) formulated in terms of the spectral theory of the Δ_M, canonically defined in terms of the metric (g_{ij}) in local coordinates by

$$\Delta_M f = \sum_{i,j} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} (\sqrt{g} g^{ij} \frac{\partial}{\partial x_j})$$

Fourier inversion is especially simple on the complete simply connected spaces of constant curvature, namely R^n, S^n, H^n. R^n has already been discussed in detail.

Fourier analysis on the sphere:

$$S^n = \{x \in R^{n+1} : \sum_{i=0}^{n} x_i^2 = 1\}.$$

Spectral theory of the Laplacian is given by the L^2 expansion in spherical harmonics:

$$f(x) = \sum_{k=0}^{\infty} c_k Y_k(x), \; \Delta Y_k + k(k+n-1)Y_k = 0, \; k = 0, 1, 2$$

$$(I.5.1)$$

Pointwise convergence: $f \in L^1(S^n)$ and define the spherical average (with respect to normalized surface measure $d\sigma$

$$\bar{f}_x(r) = \int_{\{y:d(y,x)=r\}} f(y)d\sigma(y)$$

f is said to be piecewise smooth with respect to $x \in S^n$ if $r \to \bar{f}_x(r)$ has piecewise derivatives of order $[(n+1)/2]$; the differentiability index is defined as before:

$J(f,x) = -1$ if $r \to \bar{f}_x(r)$ is discontinuous, otherwise

$$J(f;x) = \max\{j \geq 0 : r \to \bar{f}_x(r) \in C^{(k)}\}$$

Theorem I.5.1

$n = 1, 2 \Rightarrow$ (I.5.1) converges to $\bar{f}_x(0+0)$.

$n \geq 3 \Rightarrow$, (I.5.1) is convergent at $x \in S^n$ if and only if $J(f;x) \geq [(n-3)/2]$. Otherwise $J(f;x) < (n-5)/2$ and (I.5.1) diverges as $\sum_{|k|\leq M} c_k Y_k(x$ $M^{(n-5)/2-J(f;x)}$, $M \to \infty$.

Analysis on hyperbolic space: Fourier analysis on hyperbolic space is effectively discussed in terms of the representation

$$H^n = \{x \in R^{n+1} : [x, x] = 1, x_0 > 0\}$$

$$[x, y] = x_0 y_0 - \sum_{j=1}^{n} x_j y_j$$

The Fourier transform of $f \in H^n$ is

$$\widehat{f}(\mu, u) = |\frac{\Gamma(\sigma + iu)}{\Gamma(u)}|^2 \int_{H^n} f(y)[y, \xi(u)]^{-\sigma - iu} \, dy$$

and the spherical partial sum is

$$
\begin{aligned}
S_M f(x) &= \int_0^M \int_{S^{n-1}} [x, \xi(u)]^{-\sigma - i\mu} \widehat{f}(\mu, u) \, du \, d\mu \\
&= \int_0^M (\int_{H^n} \phi_\mu^n(d(x, y)) f(y) \, dy) d\mu
\end{aligned}
$$

$$\phi_\mu'' + (n-1)\coth r \phi_\mu' + (\mu^2 + \sigma^2)\phi_\mu = 0, \quad \phi(0) = 1$$

The spherical mean value, piecewise smoothness and differentiability index are defined as in the case of the sphere S^n, leading to

Theorem 5.2 Let f be piecewise smooth w.r.t. $x \in H^n$.

If $n = 1, 2$ then $\lim_M S_M f(x) = \bar{f}_x(0 + 0)$.

If $n \geq 3$ then $\lim_M S_M f(x)$ exists if and only if $J(f; x) \geq [(n - 3)/2]$. If $J(f; x) \leq (n - 5)/2$ we have divergence: $S_M f(x) \sim M^{(n-5)/2 - J(f; x)}, M \to \infty$

II.6 Eigenfunction expansions on balls

In all of the previous situations we had a manifold without boundary (R^n, T^n, S^n, H^n). The condition for convergence of the Fourier partial sums depends only on the dimension $n = 2\alpha + 2$. When we pass to manifolds with boundary we need to impose boundary conditions (Dirichlet/Neumann/etc) in order to obtained a well-defined Fourier expansion based on the Laplace operator. A prototype is the Fourier-Bessel series which correspond to the Laplaceian of Eucliudean space. In general, the convergence/divergence will depend on a second parameter β.

In general, we consider the radial Laplace operator of a Riemannian manifold with symmetry (surface of revolution):

$$ds^2 = dr^2 + g(r)^2 d\theta^2$$

where $d\theta^2$ is the metric of the standard sphere S^{n-1} and g is a smooth function with $g(0) = 0, g'(0) = 1$. The Laplace operator is written

$$\Delta_g = \frac{\partial^2}{\partial r^2} + (n-1)\frac{g'(r)}{g(r)}\frac{\partial}{\partial r} + \text{angular terms}$$

Let $V(r) = r^{n-1}g(r)$ be the volume factor, λ_m be the eigenvalues and $\phi_m(r)$ be the radial eigenfunctions:

$$\phi_m''(r) + \frac{V'(r)}{V(r)}\phi_m'(r) + \lambda_m \phi_m(r) = 0, \qquad 0 < r < a$$

$$\cos \beta \, \phi_m(a) + a \sin \beta \, \phi_m'(a) = 0$$

where $0 \le \beta < \pi$.

The Fourier expansion of $f \in L^2((0,a); \Delta(r)\, dr)$ is

$$f(r) \sim \sum_m A_m \phi_m(r), \qquad A_m = \frac{\int_0^a f(r)\phi_m(r)V(r)\, dr}{\int_0^a \phi_m(r)^2 V(r)\, dr}$$

The problem is defined by two parameters

$$(\alpha, \beta) \in [-\frac{1}{2}, \infty) \times [0, \pi)$$

One may expect that the convergence will depend on the boundary condition. For example in three dimensions ($\alpha = 1/2$) the function $f(r) \equiv 1$ has a convergent Neumann expansion ($\beta = \pi/2$) written $1 = 1$. But the Dirichlet boundary condition with $a = 1$ $\beta = 0$, leads to the series

$$1 \sim \frac{1}{2} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\sin m\pi r}{m\pi r}$$

which is not convergent at $r = 0$.

More generally, we may expect that the series (I.6.1) will converge at $r = 0$ if and only if the function f has suitable internal smoothness (depending on α) and satisfies the boundary condition in a suitably strong sense (depending on α, β).

Theorem I.6 (Bray and P, 2000) Suppose that f is piecewise smooth on $[0, a]$.

(i) Convergence in the interior If $0 < r \leq a$, the series (I.6.1) converges to $\frac{1}{2}f(r-0)+\frac{1}{2}f(r+0)$ with no further conditions.

At $r = 0$ we have the following necessary and sufficient conditions on the internal smoothness and the boundary smoothness:

(ii) Internal smoothness:

(ii a): If $\alpha < \frac{1}{2}$ no further conditions are required.

(ii b): If $k + \frac{1}{2} \leq \alpha < k + \frac{3}{2}$ we require that f have k continuous derivatives.

(iii) Boundary smoothness:

(iii a): If $\beta = 0$ and $\alpha < \frac{1}{2}$, then no further conditions are required.

(iii b): If $\beta = 0$ and $2k + \frac{1}{2} \leq \alpha < 2k + \frac{5}{2}$ we require that $f(a) = 0, \ldots L^k f(a) = 0$

(iii c): If $0 < \beta < \pi$ and $\alpha < \frac{3}{2}$, then no further conditions are required.

(iii d): If $0 < \beta < \pi$ and $2k + \frac{3}{2} \leq \alpha < 2k + \frac{7}{2}$ we require that

$$\cos \beta f_j(a) + (a \sin \beta) f_j'(a) = 0 \ f_j := L^j f, 0 \leq j \leq k$$

Interpretation: With Dirichlet boundary conditions ($\beta = 0$) there is no obstruction to convergence when the dimension is less than 3. Beyond three dimensions, for each additional four dimensions we need an extra condition at

the boundary to be satisfied in order to have convergence at the center of the ball.

With Neumann or Robin boundary conditions $(0 < \beta < \pi)$ there is no obstruction to convergence when the dimension is less than 5. Beyond five dimensions, for each additional four dimensions we need an extra condition at the boundary to be satisfied, in order to have convergence at the center of the ball.

I.7 Rank one symmetric spaces

The sphere S^n and the hyperbolic space H^n are
the prototype examples of rank one symmetric
spaces, of the compact/resp. non compact
type. In each case we can extend the previous
theory of pointwise Fourier inversion.

I.7a Compact case. A compact symmet-
ric space, written $X = G/K$, is defined by
a simply connected Lie group G with finite
center and an *isotropy subgroup* K. We ob-
tain a compact Riemanian manifold and set
$L = \mathrm{diam}(X)$. Given a reference point $o \in M$
we define the *antipodal manifold*

$$A_o = \{y \in X : d(y, o) = L\}$$

Helgason (1984, p. 167) gives the following
list:

$$
\begin{aligned}
X &= S^d & A_o &= \{ \text{ one point}\} \\
X &= P^d(R) & A_o &= P^{d-1}(R) \\
X &= P^d(C) & A_o &= P^{d-2}(C) \\
X &= P^d(H) & A_o &= P^{d-4}(C) \\
X &= P^{16}(Cay) & A_o &= S^8
\end{aligned}
$$

The Laplacian of X is written in terms of a radial part and a tangential part Δ_{S_r}:

$$
\Delta_X = \frac{\partial}{\partial r^2} + \frac{V'(r)}{V(r)}\frac{\partial}{\partial r} + \Delta_{S_r} \qquad \text{where} \quad V(r) = C\sin^p(
$$

and where $\lambda > 0$ is defined so that $2\lambda L$ is the largest root.

$f \in L^1(X)$ is K-invariant with respect to $o \in X$ if it is invariant with respect to the action of the subgroup K. We can identify f with a function on $[0, L]$ and define piecewise smoothness and $J(f; 0)$ as before

The K-invariant eigenfunctions of Δ_X are written $\phi_j(x)$. These can be identified with Jacobi polynomials. The Fourier series of a K-invariant f is written

$$f_M(x) = \sum_{j \leq M} c_j \phi_j(x) \qquad (I.7.1)$$

Theorem I.7.1 (Bray and P, 2000) Let f be piecewise smooth and K-invariant w.r.t $o \in X$. Then

(I.7.i) If $x \neq o, x \notin A_o$, then $\lim_M f_M(x)$ exists.

(I.7.ii) If $x = o$, then $\lim_M f_M(x)$ exists if and only if $J(f; o) \geq [(d-3)/2]$.

(I.7.iii) If $x \in A_o$ then $\lim_M f_M(x)$ exists as follows:

$$X = S^d \text{ requires } J(f; o) \geq [(d-3)/2]$$

$$X = P^d(R) \text{ requires } J(f; o) \geq -1$$

$$X = S^d \text{ requires } J(f; o) \geq -1$$

$$X = S^d \text{ requires } J(f; o) \geq 0$$

$$X = S^d \text{ requires } J(f; o) \geq 2$$

The proof uses a new theorem on Jacobi polynomials $y = P_n^{\alpha,\beta}(x)$, defined as the solutions of the differential equation

$$(1-x^2)y'' + ((\beta-\alpha)-(\alpha+\beta+2)x)y' + n(n+\alpha+\beta+1)y = 0$$

The expansion of a function is Jacobi series is written

$$f(x) \sim \sum_{n=-0}^{\infty} c_n P_n^{\alpha,\beta}(x), \qquad -1 \leq x \leq 1$$

$$(I.7.2)$$

Theorem I. 7.2

At an interior point $-1 < x < 1$, the Jacobi series (I.7.2) converges with no further conditions.

At the endpoint $x = 1$, the series (I.7.2) converges if and only if $J(f) > \alpha - \frac{3}{2}$

At the endpoint $x = -1$, (I.7.2) converges if and only if $J(f) > \beta - \frac{3}{2}$

I.7b Non-compact case A rank one symmetric space of the non-compact type is defined by a connected semi-simple Lie group G with finite center and a maximal compact subgroup K so that $X = G/K$.

These include real hyperbolic space, complex hyperbolic space, quaternionic hyperbolic space and the Cayley plane.

The Fourier transform on X is described in terms of the radial part of the Laplace operator (α, β determined by root space dimensions)

$$L = \frac{d}{dt^2} + ((2\alpha + 1)\coth t + (2\beta + 1)\tanh t)\frac{d}{dt}$$

and the *spherical function* ϕ_λ , which solves

$$L\phi_\lambda + (\lambda^2 + \rho^2)\phi_\lambda = 0, u(O) = 1 \qquad \rho \text{ det. by root spa}$$

The Fourier transform and spherical partial sum are given by

$$\widehat{f}(\lambda) = \int_X f(x)\phi_{-\lambda}(x)\, dx$$

$$f_M(x) = \int_0^M \widehat{f}(\lambda)\phi_\lambda(x)|c(\lambda)|^{-2}\, d\lambda \qquad \text{where}$$

$$c(\lambda) = 2^{\rho - i\lambda}\frac{\Gamma(\alpha + 1)\Gamma(i\lambda)}{\Gamma(\tfrac{1}{2}(\rho + i\lambda))\Gamma(\tfrac{1}{2}(\alpha - \beta + i + i\lambda)}$$

Theorem I.7.2 (Bray and P, 1997) Let f be a piecewise smooth K-invariant function with respect to $O \in X$. Then

(i) If $x \neq O$, then $\lim_M f_M(x)$ exists.

(ii) $\lim_M f(M(O)$ exists if and only if $J(f) \geq [\frac{dim(X) - 3}{2}]$.

The proof uses properties of the Jacobi transform, following Kornwinder (1984).

II.8 Wave equation approach

On any complete Riemannian manifold the Fourier partial sum $S_M f(x)$ can be represented in terms of the solution of the Cauchy problem for the wave equation

$$u_{tt} + \Delta u = 0, \quad u(0^+) = f, \quad u_t(0^+) = 0$$

$$(I.8.1)$$

by means of the formula

$$S_M f(x) = \int_{-\infty}^{\infty} \frac{\sin 2\pi M t}{\pi t} u(t, x) \, dt \qquad (I.8.2)$$

For example, on the real line we can write the Dirichlet formula

$$
\begin{aligned}
S_M f(x) &= \int_{-\infty}^{\infty} \frac{\sin 2\pi M t}{\pi t} f(x + t) \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin 2\pi M t}{\pi t} [f(x + t) + f(x - t)] \, dt
\end{aligned}
$$

where we identify the well-known d'Alembert formula

$$u(t, x) = \frac{f(x + t) + f(x - t)}{2}$$

In the higher dimensional case, formula (I.8.2) allows one to reduce the analysis to a one-dimensional problem, where the "angular integrations" are included in $u(t, x)$. For example, if $n = 3$, then $u(t, x) = (d/dt)[t\bar{f}_x(t)]$.

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