Properties of a supercritical superdiffusion and solutions of its corresponding differential equation

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Abstract The range and the nonextinction property of a supercritical superdiffusion and solutions of its corresponding differential equation are studied. It is proved that under a suitable condition, the conditioned superprocess of the supercritical superdiffusion is a subcritical superdiffusion.

Keywords: **supercritical superdiffusion, subcritical superdiffusion, nonlinear elliptic equation range.**

LET L be a uniformly elliptic differential operator in R^d, and $\xi = (\xi_t, \Pi_x)$ be a diffusion with the generator L. For constants A, $B > 0$, and $1 < \alpha \leqslant 2$, let

$$\psi^1(z) = -Az + Bz^\alpha; \quad \psi^2(z) = (\alpha - 1)Az + Bz^\alpha, \ z \in \mathscr{R}^1.$$

Denote by $X^1 = (X_t^1, X_\tau^1, P_\mu^1)$ the superdiffusion with parameters (L, ψ^1), and by $X^2 = (X_t^2, X_\tau^2, P_\mu^2)$ the superdiffusion with parameters (L, ψ^2). We discuss the range and the nonextinction property of supercritical superdiffusion X^1 and solutions of its corresponding differential equation. We also obtain a theorem on the relationship between X^1 and X^2.

1 Bounded solutions and singular solutions

Throghout this note D is a bounded regular domain in R^d, τ is the first exit time from D. Denote by M the set of all finite measures defined on R^d. According to ref. [1], for positive measurable function f on ∂D,

$$P_\mu^1 \exp\langle -f, X_\tau^1\rangle = \exp\langle -v, \mu\rangle, \mu \in M. \tag{1}$$

where v satisfies the integral equation

$$v(x) + \Pi_x\int_0^\tau \psi^1(v(\xi_s))\mathrm{d}s = \Pi_x f(\xi_\tau). \tag{2}$$

Theorem 1. Let f be a positive bounded continuous function on ∂D. Then

$$v_f^1(x) = -\log P_{\delta_x}^1 \exp\langle -f, X_\tau^1\rangle \tag{3}$$

defines a positive bounded solution of the following (4) and (5):

$$\mathrm{L}v - \psi^1(v) = 0, \quad \text{in } D \tag{4}$$

$$\lim_{x\in D, x\to a} v(x) = f(a), \quad \text{for all } a \in \partial D. \tag{5}$$

Proof. Let

$$z_1 = [A/(\alpha B)]^{1/(\alpha-1)}. \tag{6}$$

Then $\psi^1(z) - \psi^1(z_1) \geqslant 0$, for all $z \geqslant 0$. By (1) and (2), $v_f^1(x)$ given by (3) satisfies

$$v_f^1(x) + \Pi_x\int_0^\tau (\psi^1(v_f^1(\xi_s)) - \psi^1(z_1))\mathrm{d}s = \Pi_x f(\xi_\tau) - \psi^1(z_1)\Pi_x\tau.$$

It is easy to see that v_f^1 is bounded. By the same proof of Theorem 1.1 in ref. [2], we know that v_f^1 satisfies (4) and (5).

Lemma 1. Let $U_M = \{x : |x - x^0| < M\}$;

$$z_2 = (A/B)^{1/(\alpha-1)}; \tag{7}$$

$$u(x) = z_2 + \lambda(M^2 - r^2)^{-2/(\alpha-1)},$$

where $r = |x - x^0|$; λ is a positive constant. We have

$$u(x) \to \infty, \quad \text{as } x \to a \in \partial U_M, x \in U_M;$$

$$Lu + Au - Bu^\alpha \leqslant 0 \quad \text{in } U_M.$$

for

$$\lambda = c(1 \vee M)^{3/(\alpha-1)}, \tag{8}$$

where c is a constant depending only on α, the dimension d, L and U_M.

Lemma 2. Suppose that functions u, $v \geqslant 0$ are two times continuously differentiable in D. Then

(Ⅰ) If u, v satisfy the following conditions (i), (ii) and (iii), then $v(x) \leqslant u(x)$.

(i) $\mathrm{L}v - \psi^1(v) \geqslant \mathrm{L}u - \psi^1(u)$, in D;

(ii) $\limsup_{x\in D, x\to a}[v(x) - u(x)] \leqslant 0$ for all $a \in \partial D$;

(iii) $u(x) \geqslant z_1$, $x \in D$.

(Ⅱ) If u, v satisfy the following conditions (i′), (ii′) and (iii′), then $v(x) \geqslant u(x)$.

(i′) $\mathrm{L}v - \psi^1(v) \leqslant \mathrm{L}u$, in D;

(ii′) $\limsup_{x\in D, x\to a}[v(x) - u(x)] \geqslant 0$ for all $a \in \partial D$;

(iii′) $u(x) \leqslant z_2$.

Theorem 2.

$$v^1(x) = -\log P_{\delta_x}^1(X_\tau^1 = 0) \tag{9}$$

is a positive solution of (4) and satisfies the boundary condition

$$v^1(x) \to +\infty \text{ as } x \to a \in \partial D, x \in D.$$

Moreover,

$$v^1(x) \geqslant z_2, \text{ in } D.$$

The results of Theorem 2 follow from Theorem 1, Lemmas 1 and 2, and the arguments of Theorem 1.2 in reference [2].

Remark. By Lemma 2, if $D = U_M = \{x : |x - x^0| < M\}$, where $x^0 \in R^d$, then $v^1(x)$ defined in (9) satisfies

$$v^1(x) \leqslant z_2 + \lambda(M^2 - r^2)^{-2/(\alpha-1)}$$

with λ given by (8).

2 The range and the nonextinction property of X^1

We denote by $\mathscr{R}^1$ the range of X^1. By (1), v^1 given by (9) satisfies

$$\exp\langle - v^1, \mu\rangle = P^1_\mu(X^1_\tau = 0), \quad \text{for } \mu \in M. \tag{10}$$

Theorem 3. The range $\mathscr{R}^1$ of X^1 satisfies

$$P^1_\mu(\mathscr{R}^1 \text{ is compact}) = \exp(- z_2\mu(R^d)), \quad \text{for } \mu \in M, \tag{11}$$

where z_2 is defined by (7).

Proof. Let $D_n = \{x : |x| < n\}$, τ_n be the first exit time from D_n and

$$v^1_{(n)}(x) = - \log P^1_{\delta_x}\{X^1_{\tau_n} = 0\}, \ x \in D_n.$$

By Lemma 2.1 in ref. [2],

$$(\mathscr{R}^1 \subset \bar{D}_{n-1}) \subset (X^1_{\tau_n} = 0) \subset (\mathscr{R}^1 \subset \bar{D}_n) \quad \text{a.s.}$$

(writing "a.s." means "P^1_μ-a.s. for all $\mu \in M$"),
and therefore

$$\{X^1_{\tau_n} = 0\} \uparrow \{\mathscr{R}^1 \text{ is compact}\} \text{ a.s. as } n \uparrow \infty.$$

By (10),

$$P^1_\mu(\mathscr{R}^1 \text{ is compact}) = \lim_{n\to\infty} P^1_\mu(X^1_{\tau_n} = 0) = \lim_{n\to\infty} \exp\langle - v^1_{(n)}, \mu\rangle. \tag{12}$$

By Theorem 2 and Remark

$$z_2 \leqslant v^1_{(n)} \leqslant z_2 + c(1 \vee n)^{-3/(\alpha-1)}(n^2 - r^2)^{-2/(\alpha-1)},$$

where $r = |x|$. Letting $n \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} v^1_{(n)} = z_2$ in R^d. So (11) follows from (12).

Theorem 4. Let D_n be a sequence of bounded regular domains such that $\bar{D}_n \subset D_{n+1}$ and $D_n \uparrow R^d$, and let τ_n denote the first exit time from D_n. Then

(i) there exists a random variable Z^1 such that

$$\lim_{n\to\infty} \langle 1, X^1_{\tau_n}\rangle := Z^1 \quad \text{a.s.}$$

and the limit does not depend on the choice of D_n.

(ii) X^1 is not extinct in the sence of

$$P^1_\mu(Z^1 = 0) < 1 \quad \text{for all } \mu \in M. \tag{13}$$

Proof. By (1) and (2) we have

$$P^1_\mu \exp\langle - z_2, X^1_{\tau_n}\rangle = \exp\langle - u^1_n, \mu\rangle, \tag{14}$$

where

$$u^1_n(x) + \Pi_x\int_0^{\tau_n} \psi^1(u^1_n(\xi_s))\mathrm{d}s = z_2.$$

By Theorem 1, u^1_n is a positive bounded solution of (4) in D_n having the boundary value z_2. Since $u(x) \equiv z_2$ is a solution of (4) in R^d and $\mathrm{L}u \equiv 0$, it follows from Lemma 2 that $u^1_n(x) \equiv z_2$. So (14) can be rewritten as

$$P^1_\mu \exp\langle - z_2, X^1_{\tau_n}\rangle = \exp\langle - z_2, \mu\rangle, \tag{15}$$

by which $\exp\langle - z_2, X^1_{\tau_n}\rangle$ is a bounded martingale. By the martingale convergence theorem, $\lim_{n\to\infty}\langle 1, X^1_{\tau_n}\rangle$ exists P^1_μ-a.s. for every $\mu \in M$. Let

$$Z^1 = \lim \text{med}\langle 1, X^1_{\tau_n}\rangle.$$

Then

$$\lim_{n\to\infty}\langle 1, X^1_{\tau_n}\rangle = Z^1 \quad \text{a.s.}$$

It is easy to check that the limit does not depend on the choice of D_n.

Letting $n \to \infty$ in (15), we get

$$P_\mu^1 \exp(-z_2 \cdot Z^1) = \exp\langle -z_2, \mu\rangle,$$

which means (13) holds.

3 Connections between X^1 and X^2

Lemma 3. Let f be a bounded measurable function on ∂D. Then the following integral equation (16) has at most one positive bounded solution

$$v(x) + \Pi_x \int_0^\tau \psi^2(v(\xi_s))\mathrm{d}s = \Pi_x f(\xi_\tau), \quad x \in D. \tag{16}$$

Theorem 5. For every positive Borel function f on D,

$$P_\mu^1(\exp\langle -f, X_\tau^1\rangle / \mathscr{R}^1 \text{ is compact}) = P_\mu^2(\exp\langle -f, X_\tau^2\rangle). \tag{17}$$

Proof. Step 1. Suppose that f is bounded. Let

$$v_f^1(x) = -\log P_{\delta_x}^1 \exp\langle -f, X_\tau^1\rangle; \tag{18}$$

$$v_f^2(x) = -\log P_{\delta_x}^2 \exp\langle -f, X_\tau^2\rangle, \quad x \in D. \tag{19}$$

Then v_f^1 is a bounded solution of integral equation (2), and v_f^2 is a bounded solution of integral equation (16). It is easy to check that $v_{(f+z_2)}^1 - z_2$ also satisfies (16), where z_2 is defined by (7).

By Lemma 3,

$$v_{(f+z_2)}^1 - z_2 = v_f^2. \tag{20}$$

Using (11), (18), (19) and (20), we can obtain (17) holds for the positive bounded f.

Step 2. If f is a positive function, (17) holds for $f \wedge n$. Letting $n \to \infty$, we know that (17) holds for f.

Corollary 1. $P_\mu^1(Z^1 = 0) = \mathrm{e}^{-A/B}$; $P_\mu^1(Z^1 = \infty) = 1 - \mathrm{e}^{-A/B}$.

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