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REGULAR SOLUTIONS FOR SCHRÖDINGER EQUATION ON UNBOUNDED DOMAINS [1]

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Abstract The authors study a class of solutions,namely, regular solutions of the Schrödinger equation $(\frac{1}{2}\Delta + q)u = 0$ on unbounded domains. They definite the regular solutions in terms of sample path properties of Brownian motion and then characterize them by analytic method. In Section 4, they discuss the regular solution to the stochastic Dirichlet problem for the equation $(\frac{1}{2}\Delta + q)u = 0$ having limit α at infinity.

Key words Schrödinger equation, Regular solution, Stochastic Dirichlet problem.

1 Introduction

Let $\{X(t), t \geq 0\}$ be the Brownian motion in $R^d, d \geq 1. P_x$ and E_x denote the probability and expectation under $X(0) = x$. Let D be a domain in $R^d; \partial D = \bar{D} \cap \bar{D}^c$ the boundary of D, where $\bar{D}$ is the closure and D^c the complement of D. For any Borel set E we put

$$\tau_E = \inf\{t > 0, \quad X(t) \notin E\} \qquad (\inf \phi = \infty),$$

namely, the first exit time from E. The class of points which are regular for E will be denoted by E^r (see [6]). Let the class K_d and K_d^{loc} be defined as in [1]. For $q \in K_d^{\text{loc}}$ (if q is given only in D, then we assume $q(x) = 0$ for $x \in R^d - D$), as an abbreviation we put

$$e_q(t) = \exp \int_0^t q(X(s))ds.$$

For $f \geq 0$ on ∂D, we put for $x \in \bar{D}$:

$$u(q, f; x) = E_x(e_q(\tau_D) f(X(\tau_D)); \tau_D < \infty); \quad w_q(x) = E_x(e_q(\tau_D))$$

provided it is well-defined. The function $u(q, 1; \cdot)$ is called the gauge for (D, q) . We say that the gauge theorem holds for (D, q) if that $u(q, 1; \cdot) \not\equiv +\infty$ in D implies that $u(q, 1; \cdot)$ is bounded in D (see [4]).

Recently,many authors have been interested in the probabilistic treatment of the following Schrödinger equation:

$$\frac{1}{2}\Delta u + qu = 0, \quad \text{in} \quad D. \tag{1.1}$$

Where Δ is the Lapace operator,

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When $m(D) < \infty$ (where m denotes the Lebesgue measure) and $q \in L^\infty(D)$, Chung and Rao[2] showed that the gauge theorem holds for (D,q) and solved the problem of representing the bounded solution to the first boundary value problem for (1.1). Subsequently a large class of q was studied by Aizenman and Simon[1], which is known as the Stummel-Kato class K_d. When $m(D) = \infty$, the results about (1.1) are very little. Ren[7] showed that $E.(\varphi(X(\tau_D))e_q(\tau_D); \tau_D < \infty) + \alpha E.(e_q(\tau_D); \tau = \infty)$ is the unique bounded solution to the first boundary value problem for (1.1) with boundary value φ, under the condition that $w_q(x)$ is bounded on D (where $m(D)$ may be $+\infty$). But generally, the gauge theorem is not true for unbounded domains (see example in [9]). When the gauge theorem isn't valid for (D,q), we can not know whether the bounded solution for (1.1) exists. So in this paper, we will consider a new class of solutions for (1.1), namely, regular solutions, instead of bounded solutions.

Let $C^{(0)}(D)$ and $C^{(k)}(D), k \geq 1$, denote respectively the classes of continuous and k times continuously differentiable functions on D, and $H(D)$ the class of Hölder continuous functions (see the definition in [2]).

We say that u is a solution of (1.1), if $u \in C^2(D)$ and satisfies (1.1).

2 Some Lemmas

Lemma 2.1 Let $q \in K_d^{\mathrm{loc}}, D$ be a domain and K be a compact subset of D. there exists a constant $C > 0$ which depends only on D, K and local norms A of q such that for any $f \geq 0$ such that $u(q,f;\cdot) \not\equiv \infty$ in K, we have

$$\sup_{x\in K} u(q,f;x) \leq C \inf_{x\in K} u(q,f;x).$$

See Zhao [8].

Lemma 2.2 Let $q \in K_d^{\mathrm{loc}}, D$ be a domain and K be a compact subset of D . There exists a constant $C > 0$ which depends only on D, K and local norms A of q such that $E.(e_q(\tau_D); \tau_D = \infty) \not\equiv \infty$ in K, we have

$$\sup_{x\in K} E_x(e_q(\tau_D); \tau_D = \infty) \leq C \inf_{x\in K} E_x(e_q(\tau_D); \tau_D = \infty).$$

The proof of Lemma 2.2 is similar to that of Theorem 6 in [8].

Lemma 2.3 Let $q \in K_d^{\mathrm{loc}}$, D be a domain. If $E.(e_q(\tau_D); \tau_D < \infty) \not\equiv \infty$ on D , then for any bounded domain D_0 such that $\bar{D}_0 \subset D$, we have

$$\sup_{x\in D_0} E_x(e_q(\tau_{D_0})) < \infty.$$

Proof Let $v(x) = E_x(e_q(\tau_D); \tau_D < \infty)$ in D. For any bounded domain D_0 such that $\bar{D}_0 \subset D$, by the strong Markov property we have :

$$E_x(e_q(\tau_D)I_{(\tau_D<\infty)}/\mathcal{F}_{\tau_{D_0}}) = e_q(\tau_{D_0})v(X(\tau_{D_0})), \quad x \in D_0.$$

It follows from Lemma 2.1 that $C = \inf_{x\in\bar{D}_0} v(x) > 0$. So we have

$$\sup_{x\in D_0} E_x(e_q(\tau_{D_0})) \leq \frac{1}{C} \sup_{x\in\bar{D}_0} E_x(e_q(\tau_D); \tau_D < \infty) < \infty.$$

The last inequality follows from Lemma 2.1.

3 Regular Solutions for Schrödinger Equation

Throughout this paper, D is assumed to be fixed. The class of solutions of $\frac{1}{2}\Delta u + qu = 0$ will be denoted by S^q .

Definition 3.1 $u \in S^q$ is said to be regular on D , iff for every $x \in D$,

(I) the limit $\lim_{t\uparrow\tau_D} u(X(t))$ exists and is finite a. s. (P_x) ;

(II) the above limit is integrable with respect to P_x and

$$u(x) = E_x(\lim_{t\uparrow\tau_D} u(X(t))e_q(\tau_D))).x \in D. \tag{3.1}$$

Definition 3.2 $u \in S^q$ is said to be singular on D, iff for every $x \in D$,

$$\lim_{t\uparrow\tau_D} u(X(t)) = 0, \quad \text{a.s.} \quad (P_x). \tag{3.2}$$

We use S_r^q to denote the class of all regular solutions of $\frac{1}{2}\Delta u + qu = 0$ on D, S_s^q the class of singular solutions of $\frac{1}{2}\Delta u + qu = 0$ on D. Clearly, $S_r^q \subset S^q, S_s^q \subset S^q$, and $\{0\} = S_r^q \cap S_s^q$.

Let

$Q \triangleq \{q : q \in K_d \cap H(D)$ such that $w_q(x)$ is bounded in $D\}$,

$S_1^q \triangleq \{u \in S^q; u$ is bounded on $D\}$,

$S_2^q \triangleq \{u \in S^q; \exists q_n \in Q$ such that $q_n \uparrow q$ and $u_n \in S_1^{q_n}$ such that $u_n \uparrow u\}$,

$S_3^q \triangleq \{u \in S^q; \exists u_1, u_2 \in S_2^q$ and $a_1, a_2 \in R^1$ such that $u = a_1u_1 + a_2u_2\}$.

Remark (1) If $q \in Q$, by Theorem 3.1 bleow, we know $S_1^q \neq \phi$.

(2) For any $q \in K_d \cap H(D)$, it follows from Theorem 3.6 below that S_2^q is well defined .

(3) If $q \in Q$, then $S_2^q = \{u \in S^q, \exists u_n \in S_1^q$ such that $u_n \uparrow u\}$.

(4) Regular solutions of $\frac{1}{2}\Delta u + qu = 0$ are defined in terms of the sample path properties of Brownian motion. In the following we will show $S_r^q = S_3^q$. So regular solutions can be characterized by analytic method.

Theorem 3.1 Let $q \in K_d \cap H(D)$. Suppose $u(q, 1; \cdot) \not\equiv \infty$ in D. Then u is a solution of (1.1) iff

(I) u is locally bounded;

(II) for any bounded domain D_0 such that $\bar{D}_0 \subset D$,

$$u(x) = E_x(e_q(\tau_{D_0})u(X(\tau_{D_0}))), \quad x \in D_0. \tag{3.3}$$

Proof Suppose u satisfies (1.1) in D. Then for any bounded domain D_0 such that $\bar{D}_0 \subset D, u(x)$ is a bounded solution of $\frac{1}{2}\Delta u + qu = 0$ in D_0 , and $u(x)$ is continuous in $\bar{D}_0$. It follows from Lemma 2.3 and Theorem 2.3 in [2] that (3.3) holds. Obviously, u is locally bounded.

Conversely suppose u satisfies (I) and (II). For any bounded domain D_0 such that $\bar{D}_0 \subset D$, by Theorem 2.1 in [2] and Lemma 2.3, we have

$$\frac{1}{2}\Delta u + qu = 0 \quad \text{in} \quad D_0.$$

So $\frac{1}{2}\Delta u + qu = 0$ in D .

Before further discussions, let us introduce the concept of "Brownian motion killed outside D ". let $D_\partial = D \cup \partial$, the one-point compactification of D . Define a process $\{\tilde{X}(t), t \geq 0\}$ living on the state space D_∂ as follows

$$\tilde{X}(t) = \begin{cases} X(t) & \text{if } t < \tau_D, \\ \partial & \text{if } \tau_D \leq t \leq \infty. \end{cases}$$

We call $\tilde{X}$ the "Brownian motion killed outside D ". It follows from Theorem 4.5.2 in [3] that $\{\tilde{X}(t), \mathcal{F}_t, t \geq 0\}$ is a Hunt process, where $\mathcal{F}_t = \sigma\{X(s), s \leq t\}$.

Let

$$\tilde{q}(x) = \begin{cases} q(x), & \text{if } x \in D, \\ 0, & \text{if } x = \partial; \end{cases}$$

$$\tilde{e}_q(t) = \exp\left(\int_0^t \tilde{q}(\tilde{X}(s))ds\right), \quad t \geq 0.$$

Theorem 3.2 Let $q \in K_d \cap H(D)$ such that $w_q(\cdot) \not\equiv \infty$ in D, and let $u \in S^q$, $u \geq 0$. Then $\lim_{t \uparrow \tau_D} u(X(t))$ exists and is finite a.s. (P_x) for each $x \in D$, and $u(x) \geq E_x(e_q(\tau_D) \cdot \lim_{t \uparrow \tau_D} u(X(t))) \in S^q$.

Proof Let $u(\partial) = 0$. For any $x \in D$, by Theorem 6 in [5], we have $\{\tilde{e}_q(t)u(\tilde{X}(t)), \mathcal{F}_t, t \geq 0\}$ is a supmartingale under P_x. It then follows from Theorem 1.4.1 in [3] and its Corollary 2 that for each $x \in D$, $\lim_{t \uparrow \tau_D}(e_q(t)u(X(t)))$ exists and is finite a.s. (P_x). Since $e_q(t) > 0$ and $\lim_{t \uparrow \tau_D} e_q(t) = e_q(\tau_D) > 0$ for any $x \in D$, we have

$$\lim_{t \uparrow \tau_D} u(X(t)) = \lim_{t \uparrow \tau_D} \frac{e_q(t)u(X(t))}{e_q(t)} = \frac{\lim_{t \uparrow \tau_D}[e_q(t)u(X(t))]}{e_q(\tau_D)}, \quad \text{a.s. } (P_x).$$

Hence $\lim_{t \uparrow \tau_D} u(X(t))$ exists and is finite a.s. (P_x).

Let D_n be bounded domains such that $\bar{D}_n \subset D$ and $D_n \uparrow D$. Then

$$u = E_\cdot(e_q(\tau_{D_n})u(X(\tau_{D_n}))$$

by Theorem 3.1. Choose $x \in D$, then $P_x(\tau_{D_n} \uparrow \tau_D) = 1$. Thus by Fatou's lemma,

$$u(x) \geq E_x(\lim_{n \to \infty} e_q(\tau_{D_n})u(X(\tau_{D_n}))) = E_x(e_q(\tau_D) \lim_{t \uparrow \tau_D} u(X(t))), x \in D.$$

By the strong Markov property and Theorem 3.1 , it is easy to check that

$$E_\cdot(e_q(\tau_D) \lim_{t \uparrow \tau_D} u(X(t))) \in S^q.$$

Theorem 3.3 Let $q \in K_d \cap H(D)$ such that $w_q(\cdot) \not\equiv \infty$ in D. Then for any $x \in D$, $\lim_{t \uparrow \tau_D} w_q(X(t)) = 1$, a.s. (P_x) .

Proof Let $u(\cdot) = w_q(\cdot)$. Then $u \geq 0$ and $u \in S^q$ by Theorem 3.1. It follows from Theorem 3.2 that $\lim_{t \uparrow \tau_D} u(X(t))$ exists and is finite a.s.(P_x). Let D_n be bounded domains such that $\bar{D}_n \subset D$ and $D_n \uparrow D$. By the strong Markov property we have

$$E_x(e_q(\tau_D)/\mathcal{F}_{\tau_{D_n}}) = e_q(\tau_{D_n})u(X(\tau_{D_n})).$$

Letting $n \to \infty$ in the above equality, for any $x \in D$, we have

$$\lim_{n\to\infty} u(X(\tau_{D_n})) = 1, \text{a.s.} \ \ (P_x).$$

Since $\lim_{t\uparrow\tau_D} u(X(t))$ exists a.s. (P_x) as shown above we obtain $\lim_{t\uparrow\tau_D} u(X(t)) = 1$, a.s. (P_x).

Theorem 3.4 Let q, u be as in Theorem 3.2 except that u need not be non-negative but bounded from below. If in addition $\inf_D w_q(\cdot) > 0$, then the results of Theorem 3.2 also hold.

Proof Since $\inf_D w_q(\cdot) > 0$ and u is bounded from below, there is a constant $C > 0$, such that

$$f(x) \triangleq u(x) + C w_q(x) \geq 0, \quad x \in D.$$

By Theorem 3.1, $f(x) \in S^q$. Hence by Theorem 3.2, for any $x \in D$,

$$\lim_{t\uparrow\tau_D} f(X(t)) \quad \text{exists a.s.} \ \ (P_x) \ \ \text{and} \ \ f(x) \geq E_x(e_q(\tau_D) \lim_{t\uparrow\tau_D} f(X(t))).$$

Since $\lim_{t\uparrow\tau_D} w_q(X(t)) = 1$ by Theorem 3.3, we know $\lim_{t\uparrow\tau_D} u(X(t))$ exists a.s. (P_x), and

$$u(x) \geq E_x(e_q(\tau_D) \lim_{t\uparrow\tau_D} u(X(t))), x \in D.$$

Therefore $E.(e_q(\tau_D) \lim_{t\uparrow\tau_D} u(X(t)))$ belongs to S^q by Theorem 3.1.

Theorem 3.5 Let $q \in K_d \cap H(D)$ such that $w_q(x) \not\equiv \infty$ and $\inf_D w_q(\cdot) > 0$ in D . then $S_3^q \subset S_r^q$.

Proof Let $q_0 \in Q$ and $u \in S_1^{q_0}$. Then u is a bounded solution of (1.1) and $w_{q_0}(\cdot)$ is bounded on D. It follows from Theorem 3.4 that for any $x \in D$, $\lim_{t\uparrow\tau_D} u(X(t))$ exists a.s. (P_x) and $u(x) \geq E_x(e_{q_0}(\tau_D) \lim_{t\uparrow\tau_D} u(X(t)))$. Since $-u \in S_1^{q_0}$, we similarly have $-u(x) \geq -E_x(e_{q_0}(\tau_D) \lim_{t\uparrow\tau_D} u(X(t)))$. Hence (3.1) holds.

If $u \in S_2^q$, then there exists $q_n \in Q$ and $u_n \in S_1^{q_n}$ such that $q_n \uparrow q$ and $u_n \uparrow u$. By the above result we have

$$u_n(x) = E_x(e_{q_n}(\tau_D) \lim_{t\uparrow\tau_D} u_n(X(t))), \quad x \in D.$$

Letting $n \to \infty$, since $\lim_{t\uparrow\tau} u(X(t))$ exists a.s. (P_x) by Theorem 3.4 and $u_n \leq u$, we have

$$u(x) \leq E_x(e_q(\tau_D)) \lim_{t\uparrow\tau_D} u(X(t))), \quad x \in D.$$

On the other hand, we have again by Theorem 3.4,

$$u(x) \geq E_x(e_q(\tau_D) \lim_{t\uparrow\tau_D} u(X(t))), \quad x \in D.$$

Hence (3.1) holds for $x \in D$. Thus we have proved $S_2^q \subset S_r^q$.

By the definition of S_3^q , we easily have $S_3^q \subset S_r^q$.

Theorem 3.6 If $q \in K_d \cap H(D)$ such that $w_q(x) \not\equiv \infty$ in D, there exists $q_n \in Q$ such that $q_n \uparrow q$.

Proof Let D_n be bounded domains such that $\bar{D}_n \subset D$ and $D_n \uparrow D$. Define

$$q_n = \begin{cases} q(x) & x \in D_n \cup \{x, q(x) < 0\}, \\ 0 & x \in D_n^c \cap \{x, q(x) \geq 0\}. \end{cases}$$

Then for every $x \in D$, $q_n(x) \uparrow q(x)$. By the definitions of q_n , we can easily check $q_n \in H(D)$. By Theorem 4.5 in [1], we have $q_n \in K_d$. Let $u_n(\cdot) = E.(e_{q_n}(\tau_D))$. We are now going to show that for every fixed n, u_n is bounded in $\bar{D}$.

Since $u_n(\cdot) \le E.(e_q(\tau_D))$,by Lemma 2.1, u_n is bounded in $\bar{D}_n$. Set $\|u_n\|_{\bar{D}_n} = \sup_{\bar{D}_n} u_n(\cdot)$, and $E = D - \bar{D}_n$. Note that E is open and $\tau_E \le \tau_D$. For $x \in \bar{E}$ let us put

$$u_n^{(1)}(x) = E_x(e_{q_n}(\tau_D); \tau_E < \tau_D); \quad u_n^{(2)}(x) = E_x(e_{q_n}(\tau_D); \tau_E = \tau_D).$$

We have by the strong Markov property,

$$u_n^{(1)}(x) = E_x(e_{q_n}(\tau_E) u_n(X(\tau_E)); \tau_E < \tau_D).$$

On the set $\{\tau_E < \tau_D\}$, we have $\int_0^{\tau_E} q_n(X(t)) \le 0$ and $X(\tau_E) \in \bar{D}_n$. Hence we have

$$u_n^{(1)}(x) \le \|u_n\|_{\bar{D}_n}.$$

On the other hand, we have for $x \in \bar{E}$,

$$u_n^{(2)}(x) \le 1.$$

Combining the last two inequalities we have for $x \in \bar{D}$

$$u_n(x) \le \|u_n\|_{\bar{D}_n} + 1.$$

So we have proved that for every n, q_n belongs to Q.

Theorem 3.7 Let q belong to $K_d \cap H(D)$ such that $w_q(x) \not\equiv \infty$ in D. Then $S_3^q \supset S_r^q$.

Proof Take $u \in S_r^q$ and q_n be defined as in Theorem 3.6. Define

$$\xi^+ = (\lim_{t \uparrow \tau_D} u(X(t)) \vee 0; \quad \xi^- = (-\lim_{t \uparrow \tau_D} u(X(t))) \vee 0;$$

$$u^+ = E.(\xi^+ e_q(\tau_D)); \quad u^- = E.(\xi^- e_q(\tau_D));$$

$$\xi_n^+ = \xi^+ \wedge n; \quad \xi_n^- = \xi^- \wedge n;$$

$$u_n^+ = E.(\xi_n^+ e_{q_n}(\tau_D)); \quad u_n^- = E.(\xi_n^- e_{q_n}(\tau_D)), n = 1, 2, 3, \cdots$$

Then $u_n^+, u_n^- \in S_1^{q_n}$. Letting $n \to \infty$, it follows from the monotone convergence theorem that $u_n^+ \uparrow u^+$ and $u_n^- \uparrow u^-$ as $n \to \infty$, hence $u^+, u^- \in S_2^q$. Thus

$$u = E.(\lim_{t \uparrow \tau_D} u(X(t)) e_q(\tau_D)) = u^+ - u^- \in S_3^q.$$

This completes the proof of the theorem.

Corollary 3.8 Under the conditions of Theorem 3.5, $S_3^q = S_r^q$.

4 Stochastic Dirichlet Problem for Schrödinger Equation on Unbounded Domains

Definition 4.1 Let q, φ be functions respectively on D and ∂D , and let $\alpha \in R^1$. u is said to be a solution of the stochastic Dirichlet problem of $\frac{1}{2}\Delta u + qu = 0$ for (φ, α) , if $u \in S^q$ and for $x \in D$.

$$\lim_{t \uparrow \tau_D} u(X(t)) = \varphi(X(\tau_D)) I_{(\tau_D < \infty)} + \alpha I_{(\tau_D = \infty)}, \quad \text{a.s. } (P_x).$$

Theorem 4.1 Let $q \in K_d \cap H(D), \alpha \in R^1$ and let φ be a function on ∂D such that $\inf_D w_q(\cdot) > 0$, $u(q, |\varphi|; x) \not\equiv \infty$ and $E.(e_q(\tau_D); \tau_D = \infty) \not\equiv \infty$ on D. Then in S_3^q

$$u(\cdot) \triangleq E.(\varphi(X(\tau_D))e_q(\tau_D); \tau_D < \infty) + \alpha E.(e_q(\tau_D); \tau_D = \infty)$$

is the unique solution of the stochastic Dirichlet problem of $\frac{1}{2}\Delta u + qu = 0$ for (φ, α) .

Proof It follows from Lemma 2.1, Lemma 2.2 and Theorem 3.1 that $u \in S^q$. The proof of $u \in S_3^q$ is similar to that of Theorem 3.7. Let D_n be bounded domains such that $\bar{D}_n \subset D$ and $D_n \uparrow D$. Set $\tau_n = \tau_{D_n}, n = 1, 2, \cdots$. By the strong Markov property we have for any $x \in D_n$,

$$E_x\{[\varphi(X(\tau_D))I_{(\tau_D<\infty)} + \alpha I_{(\tau_D=\infty)}]e_q(\tau_D)/\mathcal{F}_{\tau_n}\} = e_q(\tau_n)u(X(\tau_n)).$$

Letting $n \to \infty$, we have for any $x \in D$,

$$\lim_{n\to\infty} e_q(\tau_n)u(X(\tau_n)) = [\varphi(X(\tau_D))I_{(\tau_D<\infty)} + \alpha I_{(\tau_D=\infty)}]e_q(\tau_D), \quad \text{a.s. } (P_x).$$

Hence, for any $x \in D$,

$$\lim_{n\to\infty} u(X(\tau_n)) = \varphi(X(\tau_D))I_{(\tau_D<\infty)} + \alpha I_{(\tau_D=\infty)}, \quad \text{a.s. } (P_x).$$

By the above equality and $u \in S_3^q$, we have for any $x \in D$,

$$\lim_{t\uparrow\tau_D} u(X(t)) = \varphi(X(\tau_D))I_{(\tau_D<\infty)} + \alpha I_{(\tau_D=\infty)}, \quad \text{a.s. } (P_x).$$

So u is a solution of the stochastic Dirichlet problem of $\frac{1}{2}\Delta u + qu = 0$ for (φ, α) .

The uniqueness can be easily shown by Corollary 3.8.

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