Some problems on super-diffusions and one class of nonlinear differential equations*

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Abstract The historical superprocesses are considered on bounded regular domains with a complete branching form, as a probabilistic argument, the limit property of superprocesses is studied when the domains enlarge to the whole space. As an important application of superprocess, the representation of solutions of involved differential equations is used in term of historical superprocesses. The differential equations including the existence of nonnegative solution, the closeness of solutions and probabilistic representations to the maximal and minimal solutions are discussed, which helps develop the well-known results on nonlinear differential equations.

Keywords: **diffusion process, historical superprocess, nonlinear differential equation, comparison principle, conditioned superprocess.**

1 Main results

In this paper, we assume that $L = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j(x)\frac{\partial}{\partial x_j}$ is an elliptic differential operator on R^d, and the coefficients $(a_{ij}(x))$, $(b_j(x))$ satisfy the following conditions.

(A) $a_{ij}(x) = a_{ji}(x), b_j(x)$ are bounded smooth functions on R^d, and when $\| x \| \to \infty$, $\| x \| \cdot \| b(x) \| \to 0$, where $\| x \| =: \sqrt{\sum_{i=1}^{d} x_i^2}$, $b(x) =: (b_1(x), \cdots, b_d(x))^{\mathrm{T}}$.

(B) There exists $\nu > 0$ such that, for $\forall \zeta \in R^d$, $\sum_{i,j=1}^{d} a_{ij}(x)\zeta_i\zeta_j \geqslant \nu \sum_{j=1}^{d} \zeta_j^2$, $\forall x \in R^d$.

Then L determines a Markov process of R^d denoted by $(\xi_t, \Pi_x; t \geqslant 0, x \in R^d)$, which has continuous paths and infinite lifetime[1].

We also consider such a continuous function on $R^+ = [0, \infty)$: $\Psi(z) = az + bz^2 + \int_0^\infty (\mathrm{e}^{-uz} - 1 + uz) n(\mathrm{d}u)$, where $a \in R^1$, $b \in R^+$, and $n(\mathrm{d}u)$ is a Lévy measure on $[0, \infty)$ satisfying $\int_0^\infty u \wedge u^2 \mathrm{d}u < \infty$.

Based on (ξ, Ψ), we may construct a measure-valued branching process (superprocess). To this end, we first use M_F to denote the space of all finite measures on $(R^d, \mathscr{B}(R^d))$, and let $bp\mathscr{B}(R^d)$ be the set of all bounded positive measurable functons on R^d. Write the integral of

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f w.r.t μ as $\langle \mu, f\rangle$.

Following the construction of Dynkin[2,3], there exists such a superprocess $X = \{X_t, X_{\tau_D}; P_\mu\}$ satisfying the condition that, for $\forall \mu \in M_F$, $f \in bp\mathscr{B}(R^d)$,

$$P_\mu e^{-\langle X_t, f\rangle} = e^{-\langle \mu, u(t)\rangle}, \tag{1.1}$$

where $u(t) = u(t,x)$ is the unique solution of integral equation

$$u(t,x) + \prod_x \int_0^t \Psi(u(t-s,\xi_s))\mathrm{d}s = \prod_x f(\xi_t). \tag{1.2}$$

Moreover, for any bounded regular domain D (in which the definition of regular domain is as in refs. [2,3]) on R^d, there is a corresponding random measure $X_{\tau D}$ such that

$$P_\mu e^{-\langle X_{\tau D}, \phi\rangle} = e^{-\langle \mu, u_D\rangle}, \tag{1.3}$$

in which $\tau_D = \inf\{t > 0, \xi_t \notin D\}$, $\phi \in bp\mathscr{B}(\partial D)$, and

$$u_D(x) + \prod_x \int_0^{\tau_D} \Psi(u_D(\xi_s))\mathrm{d}s = \prod_x \phi(\xi_{\tau_D}). \tag{1.4}$$

The process stated above are called historical superprocess with branching character Ψ. If $a \geqslant 0$, we call the superprocess subcritical superprocess, and if $a < 0$, supercritical.

One of the aims of this paper is to study the limit of $\{X_{\tau_D}, D \subset R^d\}$ when D is large enough, i.e. $D \uparrow R^d$. This reflects the information of extinction and non-extinction of the branching particle system and superprocess. Another aim of the paper is to apply the superprocess to represent the solutions of the involved nonlinear differential equations and then to characterize the non-negative solutions by a probabilistic technique.

Let $X \equiv X(\xi, \Psi)$ be the stated super-diffusion process, whose branching character Ψ has a general form. D and $\{D_n, n \geqslant 1\}$ are bounded regular domains on R^d. Define $z_\Psi = \inf\{z > 0, \Psi(z) > 0\}$. Then when $a \geqslant 0$, $z_\Psi = 0$; when $a < 0$, $0 < z_\Psi \leqslant \infty$, and if $z_\Psi = \infty$, $\Psi(z)$ is monotonically decreasing on z.

Theorem 1.1 *Assume that X is a historical super-diffusion process with branching character Ψ, and D is a bounded regular domain on R^d.*

1. *If $z_\Psi < \infty$, and Ψ satisfies*

$$\forall s > z_\Psi, \int_s^\infty \left(\int_s^z \Psi(t)\mathrm{d}t\right)^{-1/2} \mathrm{d}z < \infty, \tag{1.5}$$

then the non-linear differential equation

$$\begin{cases} Lu(x) = \Psi(u(x)), \ x \in D, \\ u\,|_{\partial D} = \infty \end{cases} \tag{1.6}$$

has a non-negative solution in D. Moreover the maximal and minimal solutions can be represented as $u_{\max}(x) = -\log P_{\delta_x}(X_{\tau_{D_n}} = 0$, for n large enough$)$ and $u_{\min}(x) = -\log P_{\delta_x}(X_{\tau_D} = 0)$ respectively. Here $\{D_n\}$ is a sequence of bounded regular subdomains D, $D_n \uparrow D$, but $u_{\max}$ does not depend on the choice of $\{D_n\}$.

2. *If $z_\Psi < \infty$, and Ψ does not satisfy condition (1.5), even the coefficients $(b_j(x))$ of L satisfy*

(C) $|b(x)| \leqslant \dfrac{(d-1)\nu}{2\mathrm{diam}(D)}$, $\forall x \in D$. *where* $\mathrm{diam}\,(D) =: \sup\{|x-y| : x, y \in D\}$,

then there is no non-negative solution for equation (1.6).

Theorem 1.2. *Assume that $z_\Psi < \infty$, and Ψ satisfies condition (1.5). Then the solutions of $Lu = \Psi(u)$, $u \geqslant z_\Psi$ are closed, i.e. for any bounded regular domain D on R^d, if $u_n \geqslant z_\Psi$ ($n \geqslant 1$) are solutions of the equation in D, and $u_n \to u$, then u should be a solution of the equation in D, too. Besides if the coefficients of L further satisfy condition (C), then "satisfying condition (1.5)" is necessary.*

Dynkin[2—4], Le Gall[5,6], etc. applied historical superprocesses and path-valued processes to study non-linear differential equations. However in their works, the nonlinear items of involved differential equations were such particular forms as $\Psi(z) = z^2$, or $u^{1+\beta}(0<\beta<1)$, which corresponds to the special cases of Theorems 1.1 and 1.2 with $z_\Psi = 0$, and in which Ψ obviously satisfies (1.5). We now deal with differential equations using as extensive as possible nonlinear items, and study the solutions of nonlinear differential equations using the technique of superprocess.

In the following, we describe some probabilistic features of the superprocesses.

Theorem 1.3. *Let X be a historical super-diffusion process with branching Ψ, and assume the bounded regular domains $D_n \uparrow R^d$. Then, for $\forall \mu \in M_F$, P_μ, a.s., $Y = \lim_{n\to\infty}\langle X_{\tau_{D_n}}, 1\rangle$ exists, and $P_\mu(Y=0) = 1 - P_\mu(Y=\infty) = \mathrm{e}^{-z_\Psi\langle\mu,1\rangle}(\mathrm{e}^{-\infty}\equiv 0)$. Moreover*

(i) *if $z_\Psi < \infty$, and Ψ satisfies (1.5), then $P_\mu(X_{\tau_{D_n}} = 0$, for n large enough$) = \mathrm{e}^{-z_\Psi\langle\mu,1\rangle}$;*

(ii) *if $z_\Psi < \infty$, but Ψ does not satisfy (1.5), and the coefficients $(b_j(x))$ of L satisfy condition (C), then $P_\mu(X_{\tau_{D_n}} = 0$, for n large enough$) = 0$.*

Theorem 1.4. *Assume that $z_\Psi < \infty$, and $X = (X_t, X_{\tau_D}; P_\mu)$ is a supercritical historical super-diffusion process with branching character Ψ. Then the conditioned superprocess is a subcritical superprocess. To be precise, for $\forall \mu \in M_F$, $f \in bp\mathscr{B}(R^d)$, $\phi \in bp\mathscr{B}(\partial D)$,*

$$P_\mu(\mathrm{e}^{-\langle X_t, f\rangle} \mid \lim_{t\to\infty}\langle X_t, 1\rangle = 0) = \widetilde{P}_\mu \mathrm{e}^{-\langle \widetilde{X}_t, f\rangle};$$

$$P_\mu(\mathrm{e}^{-\langle X_{\tau_D}, \phi\rangle} \mid \lim_{D\uparrow R^d}\langle X_{\tau_D}, 1\rangle = 0) = \widetilde{P}_\mu \mathrm{e}^{-\langle \widetilde{X}_{\tau_D}, \phi\rangle},$$

where $\widetilde{X} = (\widetilde{X}_t, \widetilde{X}_{\tau_D}; \widetilde{P}_\mu)$ is a super-diffusion process with the underlying process ξ and the branching character defined by

$$\widetilde{\Psi}(z) = \Psi'(z_\Psi)z + bz^2 + \int_0^\infty (\mathrm{e}^{-uz} - 1 + uz)n(\mathrm{d}u). \tag{1.7}$$

Theorem 1.4 implies that the conditioned superprocess under a suitable transformation is a superprocess, and the branching mechanism of the latter is also closely related to the original one. Actually Theorem 1.4 generalized Evens and O'connell's result on conditioned superprocesses with special branching[7].

2 Basic lemmas and technique preparations

Lemma 2.1 (comparison principle). *Assume that $z_\Psi < \infty$, and D is a bounded regular domain on R^d. u, $v \geqslant 0$ are twice differential functions in D. If u, v satisfy* (i) $Lv(x) - \Psi(v(x)) \geqslant Lu(x) - \Psi(u(x)), x \in D$; (ii) $\lim_{x\in D}, \sup_{x\to a}[v(x) - u(x)] \leqslant 0$, $\forall a \in \partial D$; (iii) $u(x) \geqslant z_\Psi$, $x \in D$. *Then for $\forall x \in D$, $v(x) \leqslant u(x)$.*

Proof. Let $w(x) = v(x) - u(x)$, if the claim is not true, then $D^+ = \{x \in D, w(x) > 0\}$ is not empty. Notice that when $z \geqslant z_\Psi$, $\Psi(z)$ is monotonically increasing on z. So for $\forall x$

$\in D^+$, $Lw(x) \geqslant \Psi(v(x)) - \Psi(u(x)) \geqslant 0$. By assumption (ii), for $\forall a \in \partial D^+$, $\lim_{x \in D}$, $\sup_{x \to a}$ $w(x) \leqslant 0$. This contradicts the maximal principle of linear differential equation. Hence the claim is true.

Next we consider the solution $u(\theta, R; x)$ of equation

$$\begin{aligned} &Lu - \Psi(u) = 0, \\ &u \mid_{\partial B(x_0, R)} = \theta, \end{aligned} \tag{2.1}$$

where $B(x_0, R)$ denotes the open Ball centered at x_0 with radius R. Let $v(x) = v(r), r = \| x - x_0 \|$. Then $Lv(x) = A(x)v''(r) + \frac{v'(r)}{r}[B(x) - A(x) + C(x)]$, where $A(x) = \frac{1}{r^2}\sum_{i,j=1}^{d} a_{ij}(x)(x_i - x_i^0)(x_j - x_j^0); B(x) = \sum_{j=1}^{d} a_{ij}(x), C(x) = \sum_{j=1}^{d}(x_j - x_j^0)b_j(x), x = (x_j)$, $x_0 = (x_j^0)$.

Suppose that $\lambda_1(x) \leqslant \lambda_2(x) \leqslant \cdots \leqslant \lambda_d(x)$ are characteristic values of matrix (a_{ij}), it follows from assumptions (A) and (B) that there exists a $\Lambda > \nu$ (Λ depending on x_0, but independent of R) such that, for $\forall x \in R^d$, (i)$\nu \in \lambda_1(x) \leqslant A(x) \leqslant \lambda_d(x) \leqslant \Lambda$; (ii) $\nu d \leqslant B(x) \leqslant \Lambda d$; (iii) $| C(x) | \leqslant (d-1)\Lambda$. So $B(x) - A(x) + C(x) \leqslant 2(d-1)\Lambda$. On the other hand, if $(b_j(x))$ satisfies (C), then $B(x) - A(x) + C(x) \geqslant \frac{\nu}{2}(d-1)$.

Let $v_1(\theta, R; r)$ be a solution of equation

$$\begin{cases} 2\Lambda\left(v''_1(r) + \frac{d-1}{r}v'_1(r)\right) = \Psi(v_1(r)), \\ v''_1(r), v'_1(r) \geqslant 0,\ v'_1(0) = 0,\ v_1(R) = \theta(\theta > z_\Psi). \end{cases} \tag{2.2}$$

Then $Lv_1 - \Psi(v_1) \leqslant 0$, and by comparison principle, $u(\theta, R; x) \leqslant v_1(\theta, R; \| x - x_0 \|), x \in B(x_0, R)$.

Let $v_2(\theta, R; r)$ be the solution of

$$\begin{cases} \frac{\nu}{2}\left(v''_2(r) + \frac{d-1}{r}v'_2(r)\right) = \Psi(v_2(r)), \\ v''_2(r),\ v'_2(r) \geqslant 0,\ v'_2(0) = 0,\ v_2(R) = \theta(\theta > z_\Psi). \end{cases} \tag{2.3}$$

Then $Lv_2 - \Psi(v_2) \geqslant 0$. Hence, by comparison principle, $u(\theta, R; x) \geqslant v_2(\theta, R; \| x - x_0 \|)$, $x \in B(x_0, R)$.

Thus under assumptions (A), (B) and (C),

$$v_2(\theta, R; \| x - x_0 \|) \leqslant u(\theta, R; x) \leqslant v_1(\theta, R; \| x - x_0 \|),\ \forall x \in B(x_0, R). \tag{2.4}$$

We are to estimate the solution of (2.1). It suffices to estimate the following equation (in which $\lambda > 0$ is constant):

$$\begin{cases} v''(r) + \frac{d-1}{r}v'(r) = \lambda\Psi(v(r)), \\ v''(r), v'(r) \geqslant 0,\ v'(0) = 0,\ v(R) = \theta(\theta > z_\Psi), \end{cases} \tag{2.5}$$

and so

$$v'(r) = \lambda r^{1-d}\int_0^r s^{d-1}\Psi(v(s))ds. \tag{2.6}$$

Since $\theta > z_\Psi$ at (2.5), by Lemma 2.1, $v(r) \geqslant z_\Psi$, $r \in [0, R]$. This implies that $v'(r) \geqslant 0$

and $v'(r) \leqslant \lambda r^{1-d}\psi(v(r))\int_0^r s^{d-1}\mathrm{d}s = \frac{\lambda r}{d}\Psi(v(r))$. Again, by (2.5), we have

$$\lambda\Psi(v(r)) \geqslant v''(r) \geqslant \frac{\lambda\Psi(v(r))}{d} \geqslant 0. \tag{2.7}$$

This shows that the conditions $v', v'' \geqslant 0$ at (2.5) are inherent rather than imposed. Furthermore, by (2.7), let $Q(v(r), v(r_0)) = \int_{v(r_0)}^{v(r)} \Psi(z)\mathrm{d}z$. Then

$$2\lambda Q(v(r), v(r_0)) \geqslant v'(r)^2 - v'(r_0)^2 \geqslant \frac{2\lambda}{d} Q(v(r), v(r_0)). \tag{2.8}$$

Since $v'(0) = 0$, $v(r) \geqslant z_\Psi$. If $v(0) > z_\Psi$, then, by (2.6), $v'(r) > 0$, $v(r) > v(0)$. Notice that if $z > z_\Psi$, $\Psi(z) > 0$. So $Q(v(r), v(0)) > 0$, and by (2.8), we have

$$\forall r \in (0, R), \frac{\sqrt{2\lambda}}{2\lambda}\int_{v(0)}^{v(r)} Q^{-1/2}(z, v(0))\mathrm{d}z \leqslant r. \tag{2.9}$$

Similarly, if there exists an $r_0 < R$ such that $v(r_0) > z_\Psi$, then $Q(v(r), v(r_0)) > 0$, and so

$$r \leqslant \frac{\sqrt{2d\lambda}}{2\lambda}\int_{v(r_0)}^{v(r)} Q^{-1/2}(z, v(r_0))\mathrm{d}z + r_0, \quad r \geqslant r_0. \tag{2.10}$$

Hence combining (2.10) with (2.4), we have

Lemma 2.2. *Assume that* $z_\Psi < \infty$, $u(\theta, R; x)$ *is a non-negative solution of* $Lu = \Psi(u)$, $u|_{\partial B(x_0, R)} = \theta$. *Then* $u(\theta, R; x) \leqslant v_1(\theta, R; \| x - x_0 \|)$, *and if* $(b_j(x))$ *satisfies* (C), $v_2(\theta, R; \| x - x_0 \| \leqslant u(\theta, R; x)$. *Besides, for all* θ, R *and* x_0, *both* $v_1(\theta, R; r)$ *and* $v_2(\theta, R; r)$ *satisfy* (2.9) *and* (2.10) $\left(\text{where } \lambda \text{ takes } \frac{1}{2\Lambda} \text{ or } \frac{2}{\nu}\right)$.

Proposition 2.1. *Assume that* $z_\Psi < \infty$, $v \geqslant z_\Psi$ *is a rotation-invariant solution of* $\Delta v = \lambda\Psi(v)(\lambda > 0)$ *on* R^d. *Then* $v \equiv z_\Psi$ *if one of the following conditions holds*: (i) v *is bounded on* R^d; (ii) Ψ *satifies condition* (1.5).

Proof. Suppose that there exists $r_0 \geqslant 0$ such that $v(r_0) > z_\Psi$.

1) If $v(r) \leqslant c(\forall r \geqslant 0)$, where c is constant. For $r > r_0$, from (2.10) it follows that $r \leqslant \frac{\sqrt{2d\lambda}}{2\lambda}\int_{v(r_0)}^{c} Q^{-1/2}(z, v(r_0))\mathrm{d}z + r_0 < \infty$. Let $r \uparrow \infty$, which leads to a contradiction.

2) If Ψ satisfies (1.5), similarly, by (2.10), for $\forall r \geqslant r_0$, $r \leqslant \frac{\sqrt{2d\lambda}}{2\lambda}\int_{v(r_0)}^{\infty} Q^{-1/2}(z, v(r_0))\mathrm{d}z + r_0 < \infty$. But when $r \to \infty$, this is impossible.

Hence, under assumption 1) or 2), $v(r) \equiv z_\Psi$, $\forall r \geqslant 0$ holds.

Lemma 2.3. *Assume that* $z_\Psi < \infty$, *and let* D *be a bounded regular domain on* R^d. f *is a non-negative bounded continuous function on* ∂D. *Then* $u(f, x) =: -\log P_{\delta_x} e^{-\langle X_{\tau_D}, f\rangle}$ *is a bounded solution of Dirichlet problem* $Lu = \Psi(u)$, $u|_{\partial D} = f$. *Moreover, if* $f \geqslant z_\Psi$, *then* $u(f, x)$ *is the unique solution satisfying* $u \geqslant z_\Psi$.

Proof. It suffices to notice that when $z_\Psi < \infty$, by (1.4), $u(f, x)$ is bounded in D. The remain is the same as in reference [2].

Lemma 2.4. *Assume that* $z_\Psi < \infty$. D *is a bounded regular domain. If* $u_n \geqslant z_\Psi (n \geqslant 1)$ *are solutions of* $Lu = \Psi(u)$ *in* D, $u_n \to u$, *and* $u_n(x)$ *are locally uniformly bounded in* D (*which means, for any compact subset* K *of* D, $u_n(x)$ *are uniformly bounded for* $x \in K$), *then* u *should be a solution of* $Lu=\Psi(u)$ *in* D.

Proof. Take a small ball $B(x_0,r)$ in D such that $\overline{B(x_0,r)}\subset D$. By the hypothesis, u_n are uniformly bounded on $\overline{B(x_0,r)}$, and for $x\in\partial B(x_0,r)$, $u_n(x)\geqslant z_\Psi(n=1,2,\cdots)$. So by Lemma 2.3, $u_n(x)$ can be expressed as

$$u_n(x)=-\log P_{\delta_x}\mathrm{e}^{-\langle X_{\tau_{B(x_0,r)}}\rangle,u_n(\cdot)},\ x\in B(x_0,r).$$

Let $n\to\infty$. Then $u(x)=-\log P_{\delta_x}\mathrm{e}^{-\langle X_{\tau_{B(x_0,r)}},u_n(\cdot)\rangle}$, $x\in B(x_0,r)$. Again, Lemma 2.3 and the arbitrariness of $\overline{B(x_0,r)}\subset D$ show that u satisfies $Lu=\Psi(u)$ in D.

Lemma 2.5. *Let* $\{D_n,\ n\geqslant 1\}$ *be a sequence of bounded regualr domains, and* $D_n\uparrow R^d$ $(n\uparrow\infty)$. *Then, under* P_μ, $\lim\limits_{n\to\infty}\langle X_{\tau_{D_n}},1\rangle$ *exists; moreover the limit does not depend on the choice of* $\{D_n,\ n\geqslant 1\}$.

Proof. Applying the special Markov property, we see that for $\forall\mu\in M_F$, when $a\geqslant 0$, $\{\mathrm{e}^{-\langle X_{\tau_{D_n}},1\rangle},n\geqslant,\ P_\mu\}$ is a bounded submartingale; when $a<0$, $z_\Psi=\infty$, the random sequence is positive supermartingale; and martingale when $a<0$, and $z_\Psi<\infty$. So in any case, $\lim\limits_{n\to\infty}\langle X_{\tau_{D_n}},1\rangle$ a. s. exists. Assume that $\{\bar D_l,\ l\geqslant 1\}$ is another sequence of bounded regular domains with $\bar D_l\uparrow R^d(l\uparrow\infty)$. Then we take a subsequence $\{D_{n_k}\}$ of $\{D_n\}$ and a subsequence $\{\bar D_{l_m}\}$ of $\{\bar D_l\}$ such that $D_{k_1}\subset\bar D_{l_1}\subset D_{k_2}\subset\bar D_{l_2}\subset\cdots$. Then the procedure above actually implies that $\{\langle X_{\tau_{D_n}},1\rangle,n\geqslant 1\}$ and $\{\langle X_{\tau_{\bar D_n}},1\rangle,\ n\geqslant 1\}$ have the same limit (w.r.t. P_μ a.s.).

3 The proof of Main Theorems

Proof of Theorem 1.1. (1) If $z_\Psi<\infty$, let $u_D(\theta,x)$ be a non-negative solution of $Lu=\Psi(u),u|_{\partial D}=\theta(\theta>z_\Psi)$. It follows from comparison principle that $u_D(\theta,\cdot)$ is increasing on θ. And define $u_D(x)=:\lim\limits_{\theta\to\infty}u_D(\theta,x)$.

On the one hand, let $v_1(\theta,R;r)$ be the solution of (2.2) and define $v_1(R,r)=\lim\limits_{\theta\to\infty}v_1(\theta,R;r)$. For given $r_0\in[0,R)$, $r\in[r_0,R)$, it follows from (2.10) that $r\leqslant\sqrt{\mathrm{d}\Lambda}\int_0^\infty\mathrm{d}z\left(\int_0^z\mathrm{d}t\Psi(t+v_1(\theta,R;r_0))\right)^{-1/2}+r^0=:p(\theta)$. $v_1(R,r_0)=\infty$, and condition (1.5) implies $p(\theta)\to r_0$; this leads to a contradiction. So $v_1(R,r_0)<\infty$, and the arbitraritness of r_0 shows $v_1(R,r)<\infty,r\in[0,R)$.

On the other hand, let $z_0\in D$, $\bar r>0$, $\overline{B(x_0,r)}\subset D$. Since $u_D(\theta,x)$ is a non-negative solution of $Lu=\Psi(u),u|_{\partial D}=\theta(\theta>z_\Psi)$, there exists a constant $M_{x_0,\bar r}(\theta)>0$ such that $u_D(\theta,x)|_{\partial B(x_0,\bar r)}\leqslant M_{x_0,\bar r}(\theta)=:\bar\theta$. So comparison principle implies that for $\forall x\in\overline{B(x_0,r/2)}$, $\theta>z_\Psi$, $u_D(\theta,x)\leqslant v_1(\bar\theta,\bar r;\|x-x_0\|)\leqslant\lim\limits_{\theta\to\infty}v_1(\theta,\bar r;\bar r/2)=v_1(\bar r;\bar r/2)<\infty$. Applying Lemma 2.4, $u_D(x)=\lim\limits_{\theta\to\infty}u_D(\theta,x)$ satisfies $Lu=\Psi(u)$ in $B(x_0,\bar r/2)$. The arbitrairiness of $B(x_0,\bar r)\subset D$ shows that $Lu_D=\Psi(u_D)$ in D. To check the boundary condition, it suffices to notice that, for $\forall a\in\partial D$,

$$\lim_{x\to a,x\in D}u_D(x)=\lim_{x\to a,x\in D}\lim_{\theta\to\infty}u_D(\theta,x)\geqslant\liminf_{\theta\to\infty}\lim_{x\to a,x\in D}u_D(\theta,x)=\infty.$$

We actually show that u_D is a non-negative solution of (1.6).

Let $v_D(x)$ be a non-negative solution of (1.6). Then, by comparison principle, $\forall x \in D$, $u_D(\theta,x) \leqslant v_D(x)$ $(\theta > z_\Psi)$, and $0 \leqslant u_D(x) = \lim_{\theta\to\infty} u_D(\theta,x) \leqslant v_D(x)$, $x \in D$. This implies that u_D is the minimal non-negative solution of (1.6). Moreover, by Lemma 2.3, $u_D(\theta,x) = -\log P_{\delta_x} e^{-\langle X_{\tau_D},\theta\rangle}$ $(\theta > z_\Psi, x \in D)$. So $u_{\min}(x) = u_D(x) = \lim_{\theta\to\infty}[-\log P_{\delta_x} e^{-\langle X_{\tau_D},\theta\rangle}] = -\log P_{\delta_x}[X_{\tau_D} = 0]$.

On the other hand, let $\{D_n,\ n \geqslant 1\}$ be increasing regular domains satisfying $\bar{D}_n \subset D(n \geqslant 1)$. In a similar way, applying Lemma 2.4 we can show that $u_{D_n}(x) = -\log P_{\delta_x}[X_{\tau_{D_n}} = 0]$ satisfies $Lu = \Psi(u)$, $u|_{\partial D_n} = \infty$. Hence, by comparison principle, $\forall x \in D_n$, $v_D(x) \leqslant u_{D_n}(x)$, $n = 1,2,\cdots$. Applying the special Markov property of superprocesses, we have $\{X_{\tau_{D_n}} = 0\} \subseteq \{X_{\tau_{D_{n+1}}} = 0\}$ (where $\bar{D}_n \subset D_{n+1}$, $n \geqslant 1$), namely $\{u_{D_n}(x),\ n \geqslant 1\}$ is decreasing on n. So $\forall x \in D$, $v_D(x) \leqslant \lim_{n\to\infty} u_{D_n}(x) = -\log P_{\delta_x}(\{X_{\tau_{D_n}} = 0$, for n large enough$\})$, which implies that $u_{\max}(x) =: -\log P_{\delta_x}(\{X_{\tau_{D_n}} = 0$, for n large enough$\}) \geqslant v_D(x)$.

We are to show that $u_{\max}$ satisfies (1.6). For any given $N \geqslant 1$, if $n \geqslant N$, $u_{D_n}(x)$ satisfy $Lu = \Psi(u)$ in D_N, and $u_{D_n}(x) \leqslant u_{D_N}(x), x \in D_N$. Hence, for $x \in D_N$, Lemma 2.4 implies that $u_{\max}(x) = \lim_{n\to\infty} u_{D_n}(x)$ is also a solution of $Lu = \Psi(u)$ in D_N. Let $D_N \uparrow D$. Then $u_{\max}(x)$ satisfies $Lu = \Psi(u)$ in D. And for $a \in \partial D$, $\lim_{x\to a, x\in D} u_{\max}(x) \geqslant \lim_{x\in a, x\in D} v_D(x) = \infty$. This shows that $u_{\max}(x)$ is the maximal solution of (1.6).

(2) Assume that $z_\Psi < \infty$, Ψ does not satisfy (1.5), and $(b_j(x))$ satisfies condition (C). If (1.6) has a non-negative solution $v_D(x)$, then from the proof of (1) it follows that, $v_D(x) \geqslant -\log P_{\delta_x}(X_{\tau_D} = 0)$. But the latter may not be a solution of (1.6). Take $B(x_0,R)$ such that $\bar{D} \subset B(x_0,R)$. Then special Markov property implies that, P_{δ_x} a.s. $(X_{\tau_D} = 0) \subseteq (X_{\tau_{B(x_0,r)}} = 0)$, and $v_D(x) \geqslant -\log P_{\delta_x}(X_{\tau_{B(x_0,R)}} = 0) =: u_R(x)$. Let $u(\theta,R;x)$ be a solution of (2.1). If $\theta > z_\Psi$, comparison principle implies that $u_R(x) \geqslant u(\theta,R;x), x \in B(x_0,R)$. But, by Lemma 2.2, $u(\theta,R;x) \geqslant v_2(\theta,R;\|x-x_0\|)$. So $u_R(x) \geqslant v_2(\theta,R;\|x-x_0\|)(x \in B(x_0,R), \theta > z_\Psi)$, where v_2 is the solution of (2.5) $(\lambda = 2/\nu)$.

In the following we are ready to show that if $\theta \uparrow \infty$, then $v_2(\theta,R;r) \uparrow \infty$. Since $v_2(\theta,R;r)$ is increasing on r, it suffices to show $\lim_{\theta\to\infty} v_2(\theta,R;0) = \infty$. Otherwise, suppose that $v_2(\theta,R;0) \uparrow c < \infty$ (when $\theta \uparrow \infty$). Let $\lambda = 2/\nu$ at (2.9), and $r \to R$. Then $\int_{v_2(\theta,R;0)}^{\theta} \left[\int_{v_2(\theta,R;0)}^{z} \Psi(s)\mathrm{d}s\right]^{-1/2} \mathrm{d}z \leqslant \frac{2}{\sqrt{v}} R$. And letting $\theta \to \infty$, we have

$$\int_c^\infty \left[\int_c^z \Psi(s)\mathrm{d}s\right]^{-1/2} \mathrm{d}z \leqslant 2R/\sqrt{\nu} < \infty. \tag{3.1}$$

Notice that, when $\theta > z_\Psi$, $v_2(\theta,R;r) \geqslant z_\Psi$. We claim that $c = \lim_{\theta\to\infty} v_2(\theta,R;0) > z_\Psi$. Otherwise, if $c = z_\Psi$, then $v_2(\theta,R;0) \equiv z_\Psi$, $\forall \theta > z_\Psi$. But applying historical super-Brownian motion $\bar{X} = (\bar{X}_t, \bar{X}_\tau, \bar{P}_\mu)$, whose branching character $\Psi(z) = 1/\nu\Psi(z)$, to solve differential equationis $\frac{1}{2}\Delta v = \Psi(v)$, $v|_{\partial B(x_0,R)} = \theta$ (which is eq. (2.3)), we have $v(x) = v_2(\theta,R;$

$\| x - x_0 \|) = -\log P_{\delta_x} \mathrm{e}^{-\langle \bar{X}_{\tau_{B(x_0,R)}}, \theta \rangle}$. i.e. $v_2(\theta, R; 0) = -\log P_{\delta_{x_0}} \mathrm{e}^{-\langle \bar{X}_{\tau B(x_0,R)}, \theta \rangle}$. If $v_2(\theta, R; 0) \equiv z_\Psi$, then $P_{\delta_{x_0}} [\bar{X}_{\tau_{B(x_0,R)}} = 0] = 1$. But by a differential operation of (1.3) and (1.4), we have $P_\mu \langle X_{\tau_D}, \phi \rangle = \langle \mu, \prod_x \phi(\xi_{\tau_D}) \rangle$. Particularly, $\bar{P}_{\delta_{x_0}} \langle \bar{X}_{\tau_{B(x_0,R)}}, 1 \rangle = \langle \delta_{x_0}, 1 \rangle = 1$. This leads to a contradiction, so $c > z_\Psi$.

On the other hand, since $\Psi(s)(s > z_\Psi)$ is a strictly positive continuous function, condition (1.5) is actually equivalent to the statement that there exists a constant $s_0 > z_\Psi$ such that $\int_{s_0}^{\infty} \left[\int_{s_0}^{z} \Psi(s) \mathrm{d}s \right]^{-1/2} \mathrm{d}z < \infty$. So (3.1) implies (1.5). But this contradicts the assumption of theorem. So under the assumption of (2) there is no non-negative solution for (1.6).

Proof of Theorem 1.2. Let $u_n(x) \geqslant z_\Psi$, $n \geqslant 1$ be solutions of (1.6) (where D is a bounded regular domain), $u_n(x) \to u(x)(n \to \infty)$. For $\forall x \in D$, take a small ball $B(x_0, \bar{r})$ such that $\overline{B(x_0, \bar{r})} \subset D$. By Theorem 1.1 and comparison principle, $u_n(x) \leqslant -\log P_{\delta_x}(X_{\tau_{B(x_0,\bar{r})}} = 0)$, $x \in B(x_0, \bar{r})$. So $u_n(x)$ are locally uniformly bounded in $B(x_0, \bar{r})$, and by Lemma 2.4, $u(x)$ satisfies $Lu = \Psi(u)$ in $B(x_0, \bar{r})$, which means that u is a solution of the equation. So the solutions of equation are closed if $z_\Psi < \infty$ and Ψ satisfies condition (1.5).

Next we show that when the coefficients of L satisfy (C), "Ψ satisfying (1.5)" is necessary for the closeness of solutions. To do so, we give an example, in which Ψ does not satisfy (1.5) and the solutions are not closed. In fact, let $u_D(k, x)$ be the solutions of $Lu = \Psi(u)$, $u|_{\partial D} = k(k > z_\Psi)$. It follows from the proof of Theorem 1.1(2) that if Ψ does not satisfy (1.5), $\lim_{k \to \infty} u_D(k, x) = -\log P_{\delta_x}(X_{\tau_D} = 0) \equiv \infty$, and it is not a solution of $Lu = \Psi(u)$.

Proof of Theorem 1.3. By Lemma 2.5, $Y = \lim_{D_n \uparrow R^d} \langle X_{\tau_{D_n}}, 1 \rangle$ a.s. exists, and does not depend on the choice of $\{D_n, n \geqslant 1\}$. In the following, we let $D_n = B(0, n)$.

(1) If $z_\Psi < \infty$, then $\Psi(z)$ is decreasing on z. It follows from (1.3) and (1.4) that

$$E_\mu \mathrm{e}^{-Y} = \lim_{n \to \infty} E_\mu \mathrm{e}^{-\langle X_{\tau_{B(0,n)}}, 1 \rangle} \leqslant \lim_{n \to \infty} \exp\left\{ -\int [1 - \Psi(1) \prod_x \tau_{B(0,n)}] \mu(\mathrm{d}x) \right\} = 0,$$

i.e. $P_\mu(Y = \infty) = 1$.

(2) If $z_\Psi < \infty$, let $u_n(\theta, x)(\theta > z_\Psi)$ be the unique bounded solution of $Lu = \Psi(u)$, $u|_{\partial B(0,n)} = \theta$. Then $u_n(\theta, x) \geqslant z_\Psi$, and $E_\mu \mathrm{e}^{-\theta Y} = \lim_{n \to \infty} E_\mu \mathrm{e}^{-\theta \langle X_{\tau_{B(0,n)}}, 1 \rangle} = \lim_{n \to \infty} \mathrm{e}^{-\langle \mu, u_n(\theta, \cdot) \rangle} (\theta > z_\Psi)$. On the one hand, let $v_1(\theta, n; r)$ be the solution of (2.2) (where $x_0 = 0$, $R = n$), which also satisfies $\Delta v = \frac{1}{2A} \Psi(v)$, $v|_{\partial B(0,n)} = \theta$. Noticing that, for $\theta > z_\Psi$, $z_\Psi \leqslant v_1(\theta, n; r) \leqslant \theta$, and $v_1(\theta, n; \| x \|)$ is decreasing on n, we claim that $\lim_{n \to \infty} v_1(\theta, n; \| x \|)$ is a bounded solution of $\Delta v = \frac{1}{2A} \Psi(v)$ in R^d. To show this, take an integer N, if $n \geqslant N$, $v_1(\theta, n; \| x \|)$ is a solution of $\Delta v = \frac{1}{2A} \Psi(v)$ in $B(0, N)$. Lemma 2.4 shows that $\lim_{n \to \infty} v_1(\theta, n; \| x \|)$ is also a solution of $\Delta v = \frac{1}{2A} \Psi(v)$ in $B(0, N)$, and the arbitrarity of $N \geqslant 1$ implies that it is even a solution on the whole space. So by Proposition 2.1, $\lim_{n \to \infty} v_1(\theta, n; r) \equiv z_\Psi$. But it follows from com-

parison principle that $z_\Psi \leqslant u_n(\theta,x) \leqslant v_1(\theta,n;\|x\|)$, and $\lim_{n\to\infty} u_n(\theta,x) \equiv z_\Psi$. So $E_\mu e^{-\theta Y} = e^{-z_\Psi\langle\mu,1\rangle}(\forall\theta > z_\Psi)$. Letting $\theta\uparrow\infty$, we get $P_\mu(Y=0) = e^{-z_\Psi\langle\mu,1\rangle}$. Again let $\theta\downarrow z_\Psi$. Then $E_\mu e^{-z_\Psi Y} = E_\mu e^{-\theta Y}$. Hence

$$\begin{aligned} &P_\mu[Y=0] + P_\mu[e^{-z_\Psi Y}, 0 < Y < \infty] \\ &= P_\mu[Y=0] + P_\mu[e^{-\theta Y}, 0<Y<\infty]\ (\forall\theta > z_\Psi), \end{aligned}$$

whic implies $P_\mu[0 < Y < \infty] = 0$ and $P_\mu[Y=0] = 1 - P_\mu[Y=\infty] = e^{-z_\Psi\langle\mu,1\rangle}$.

(i) Assume that $z_\Psi < \infty$, and Ψ satisfies (1.5). By special Markov property, $\{X_{\tau_{B(0,n)}} = 0\} \subseteq \{X_{\tau_{B(0,n+1)}} = 0\}(n\geqslant 1)$, and so

$$P_\mu[X_{\tau_{B(0,n)}} = 0,\ \text{for } n \text{ large enough}] = E_\mu \lim_{n\to\infty}\lim_{\theta\to\infty} e^{-\langle X_{\tau_{B(0,n)}},\theta\rangle} = \lim_{n\to\infty}\lim_{\theta\to\infty} e^{-\langle\mu,u_n(\theta,\cdot)\rangle}.$$

On the other hand, let $v_1(\infty,n;r) = \lim_{\theta\to\infty} v_1(\theta,n;r)$, where $v_1(\theta,n;r)$ is the solution of (2.2) with $R=n$. From the proof of Theorem 1.1, it follows that $v_1(x) = v_1(\infty,n;\|x\|)$ satisfies $\Delta v = \frac{1}{2\Lambda}\Psi(v)$, $v\,|_{\partial B(0,n)} = \infty$. Since $v_1(\infty,n;r)$ is decreasing on n, $\lim_{n\to\infty} v_1(\infty,n;\|x\|)$ exists, and a similar way as (2) shows that it is a solution of $\Delta = \frac{1}{2\Lambda}\Psi(v)$ in R^d. Furthermore, by Proposition 2.1, we have $\lim_{n\to\infty} v_1(\infty,n;\|x\|) \equiv z_\Psi$. Hence comparison principle implies that $z_\Psi \leqslant u_n(\theta,x) \leqslant v_1(\infty,n;\|x\|)(x\in B(0,n))$, and so $\lim_{n\to\infty}\lim_{\theta\to\infty} u_n(\theta,x) \equiv z_\Psi$. This shows that $P_\mu[X_{\tau_{B(0,n)}} = 0$, for n large enough$] = e^{-z_\Psi\langle\mu,1\rangle}$.

(ii) In the proof of Theorem 1.1, it is proved that, if $z_\Psi<\infty$ and the coefficients of L satisfy (C), and if Ψ does not satisfy (1.5), then $P_{\delta_x}(X_{\tau_D} = 0) = 0$, $x\in D$, for any bounded regular domain D. So, by special Markov property, we have $P_\mu[$for n large enough, $X_{\tau_{B(0,n)}} = 0] = \lim_{n\to\infty} P_\mu[X_{\tau_{B(0,n)}} = 0] = 0$, and the proof of Theorem 1.3 is completed.

Proof of Theorem 1.4. For $\mu\in M_F$, $\phi\in bp\mathscr{B}$, we have $P_\mu(e^{-\langle X_{\tau_D},\phi\rangle} \mid \lim_{D_n\uparrow R^d}\langle X_{\tau_{D_n}},1\rangle 0)$

$$= e^{z_\Psi\langle\mu,1\rangle} P_\mu\left(e^{-\langle X_{\tau_D},\phi\rangle} Px_{\tau_D}\left(\lim_{D_n\uparrow R^d}\langle X_{\tau_{D_n}},1\rangle = 0\right)\right) = e^{-\langle\mu,u(\phi+z_\Psi,\cdot)-z_\Psi\rangle},$$

where $u(\phi+z_\Psi,\cdot)$ is the unique positive solution of $u(\phi+z_\Psi,x) + \prod_x\int_0^{\tau_D}\Psi(u(\phi+z_\Psi,\xi_s))ds = \prod_x\phi(\xi_{\tau_D}) + z_\Psi$.

Let $w(x) = u(\phi+z_\Psi,x) - z_\Psi \geqslant 0$. Then $w(x) + \prod_x\int_0^{\tau_D}\Psi(w(\xi_s)+z_\Psi)ds = \prod_x\phi(\xi_{\tau_D})$. Defining $\widetilde{\Psi}(z) =: \Psi(z+z_\Psi)(z\geqslant 0)$, we may check that it has a form as (1.7), and $w(x)$ satisfies

$$w(x) + \prod_x\int_0^{\tau_D}\widetilde{\Psi}(w(\xi_s))ds = \prod_x\phi(\xi_{\tau_D}).$$

Lemma 2.3 implies that $w(x)$ is a bounded non-negative solution of $Lu = \widetilde{\Psi}(u)$, $u\,|_{\partial D} = \phi$, and so is $-\log\widetilde{P}_{\delta_x} e^{-\langle\widetilde{X}_{\tau_D},\phi\rangle}$. So by comparison principle, we have $w(x) = -\log\widetilde{P}_{\delta_x} e^{-\langle\widetilde{X}_{\tau_D},\phi\rangle}$. Hence $P_\mu(e^{-\langle X_{\tau_D},\phi\rangle} \mid \lim_{n\to\infty}\langle X_{\tau_{D_n}},1\rangle = 0) = \widetilde{P}_\mu e^{-\langle\widetilde{X}_{\tau_D},\phi\rangle}$. In the same way we can show that $P_\mu(e^{-\langle X_t,f\rangle} \mid \lim_{t\to\infty}\langle X_t,1 = 0\rangle = \widetilde{P}_\mu e^{-\langle\widetilde{X}_t,f\rangle})$.

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