补充例题选讲—顺序统计量

例. $X_1, \cdots, X_n$ 相互独立, $X_i \sim \mathrm{Exp}(\lambda_i), \forall i$.

- 令$Y = \min\limits_{1 \leqslant i \leqslant n} X_i =: X_I, \quad \mu = \lambda_1 + \cdots + \lambda_n$. 则

$$Y \sim \mathrm{Exp}(\mu), \quad P(I = i) = \frac{\lambda_i}{\mu}, \ \forall i, \quad Y, I \text{ 相互独立}.$$

- $P(Y > x, I = i) = P(\underbrace{x < X_i < X_j, \forall j \neq i}_{})$.
- $P(\underset{\sim}{**} | X_i = y) = P(X_j > y, \forall j \neq i) = \prod_{j \neq i} e^{-\lambda_j y}$.
- $\star\star = \int_x^\infty \lambda_i e^{-\lambda_i y} e^{-(\mu - \lambda_i) y} dy = \frac{\lambda_i}{\mu} \cdot e^{-\mu x}$.
- 注: 特别地, 若i.i.d., 则I 服从均匀分布.

例. 顺序统计量(P167). 设$X = X_1, \cdots, X_n$ i.i.d., 密度为$p(x)$.

- 从小到大排序: $X_{(1)} < \cdots < X_{(n)}$, $X_{I_1} < \cdots < X_{I_n}$.
- $\xi := (X_{(1)}, \cdots, X_{(n)})$ 的联合密度$\hat{p}(x_1, \cdots, x_n)$ 如下:

$$\hat{p}(\vec{x}) = n! \cdot p(x_1) \cdots p(x_n), \quad x_1 < \cdots < x_n.$$

- 对称性: $\vec{I} = (I_1, \cdots, I_n) \sim$ 全排中的均匀分布.
- ξ 与$\vec{I}$ 相互独立: $x_1 < \cdots < x_n$, $0 < \Delta x_r < x_{r+1} - x_r$

$$P\Big(x_r < X_{(r)} \leqslant x_r + \Delta x_r, \forall r;\ \vec{I} = (i_1, \cdots, i_n)\Big)$$

$$= P(x_r < X_{i_r} \leqslant x_r + \Delta x_r, \forall r) = \prod_{r=1}^{n} P(x_r < X < x_r + \Delta x_r)$$

$$= P\Big(x_r < X_{(r)} \leqslant x_r + \Delta x_r, \forall r\Big) \cdot \frac{1}{n!}.$$

- $X_{(1)}$ 的密度$p_1(x)$: X 的尾分布函数记为$G(x)$, 则

$$G_1(x) = P(X_{(1)} > x) = P(X_r > x, r = 1, \cdots, n) = G(x)^n$$
$$\Rightarrow p_n(x) = nG(x)^{n-1}p(x).$$

- $X_{(n)}$ 的密度$p_n(x)$: X 的分布函数记为$F(x)$, 则

$$F_n(x) = P(X_{(n)} \leqslant x) = P(X_r \leqslant x, r = 1, \cdots, n) = F(x)^n$$
$$\Rightarrow p_n(x) = nF(x)^{n-1}p(x).$$

- 下设$1 < r < n$. 往求$X_{(r)}$ 的密度.

- $X_{(r)}$ 的密度$p_r(x)$: $X_{(1)} < \cdots < X_{(n)} < X_{(n+1)} := \infty$,

$$F_r(x) = \sum_{s=r}^{n} P(X_{(s)} \leqslant x < X_{(s+1)}) = \sum_{s=r}^{n} C_n^s F(x)^s (1 - F(x))^{n-s},$$

$$\Rightarrow p_r(x) = f'(F(x))p(x), \quad f(y) = \sum_{s=r}^{n} C_n^s y^s (1-y)^{n-s}.$$

- $f'(y)$: $\frac{n!}{s!(n-s)!} s = \frac{n!}{(s-1)!(n-s)!} = nC_{n-1}^{s-1}$, $C_n^s(n-s) = nC_{n-1}^s$,

$$f'(y) = \sum_{s=r}^{n} C_n^s s y^{s-1}(1-y)^{n-s} - \sum_{s=r}^{n-1} C_n^s (n-s) y^s (1-y)^{n-s-1}$$

$$= n \sum_{s=r-1}^{n-1} C_{n-1}^s y^s (1-y)^{n-s-1} - n \sum_{s=r}^{n-1} C_{n-1}^s y^s (1-y)^{n-s-1}$$

$$= nC_{n-1}^{r-1} y^{r-1} (1-y)^{n-r}.$$

- 注: $r = 1$ 或n 时, 也成立.

例. 假设$X_1, X_2, \cdots$ i.i.d, $M_n = \max\limits_{1\leqslant i\leqslant n} X_i$.

- $M_n \nearrow M_\infty$. 则$M_\infty \overset{\text{a.s.}}{=} x_0 := \sup\{x : F(x) < 1\}$.
- 若$x_0 < \infty$, 则$F(x_0) = 1$;

$$P(M_\infty \leqslant x) \leqslant P(M_n \leqslant x) = F(x)^n \to 0, \quad \forall x < x_0.$$

- 设$x_0 = \infty$. 取$z_n \to \infty$ 使得

$$P(M_n \leqslant z_n) = (1 - G(z_n))^n \to c > 0, \quad G(x) = P(X > x).$$

- $G(z_n) = \frac{1}{n}$ 或 $\propto \frac{1}{n}$. 例,

$$G(z_n) = \frac{\lambda}{n} \Rightarrow (1 - G(z_n))^n = \left(1 - \frac{\lambda}{n}\right)^n \to e^{-\lambda}.$$

- $X \sim \mathrm{Exp}(1)$, $G(x) = e^{-x}$, $\forall x > 0$.
- 取

$$z_n = -\ln \frac{y}{n}, \quad G(z_n) = \frac{y}{n}.$$

- 则

$$P(M_n \leqslant -\ln \frac{y}{n}) = (1 - \frac{y}{n})^n \to e^{-y}.$$

- $x = -\ln y$, $y = e^{-x}$: 重指数(Gumbel)分布,

$$P(M_n - \ln n \leqslant x) \to \exp\left\{-e^{-x}\right\}, \quad \forall x.$$

例(小数定律). 设事件$A_1, \cdots, A_n$ 相互独立. 记

$$X_i = 1_{A_i}, \quad X = \sum_{i=1}^{n} X_i.$$

- 记$p_i = P(A_i)$, $\lambda = \sum_{i=1}^{n} p_i$, $q = \max_{1 \leqslant i \leqslant n} p_i$. 则

$$\sum_{k=0}^{\infty} \left| P(X = k) - \frac{\lambda^k}{k!} e^{-\lambda} \right| \leqslant 2\lambda q.$$

- 注: 特别地, 固定λ, 令$n \to \infty$, $p_i = \lambda/n$, 则$B(n, p) \to P(\lambda)$:

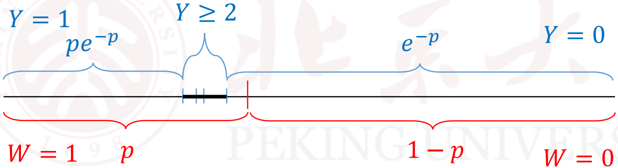
$$2\lambda q = 2\lambda^2/n \to 0.$$

- $\mathcal{L}(X) = \mathcal{L}(X_1) * \cdots * \mathcal{L}(X_n)$, $P(\lambda) = P(p_1) * \cdots * P(p_n)$.

- 比较$B(1,p)$ 与$P(p)$ 的分布列:

$$d_p = \left|(1-p) - e^{-p}\right| + \left|p - pe^{-p}\right| + \sum_{k\geqslant 2}\left|0 - \frac{p^k}{k!}e^{-p}\right|.$$

- 耦合: 定义$W := f_p(U) \sim B(1,p)$ 与$Y := g_p(U) \sim P(p)$:



- 于是, $d_p = 2P(W \neq Y)$. $1 - p \leqslant e^{-p} \Rightarrow 1 - e^{-p} \leqslant p$, 故

$$P(W \neq Y) = 1 - \left(pe^{-p} + (1-p)\right) = p - pe^{-p} \leqslant p^2.$$

- $U_1, \cdots, U_n$ i.i.d. $\sim U(0,1)$. 取$W_i = f_{p_i}(U_i)$, $Y_i = g_{p_i}(U_i)$.
- $W_1, \cdots, W_n$ 相互独立; $Y_1, \cdots, Y_n$ 相互独立:

$$W = \sum_{i=1}^{n} W_i \overset{d}{=} X, \quad Y = \sum_{i=1}^{n} Y_i \sim P(\lambda).$$

- $P(W \neq Y) \leqslant \sum_i P(W_i \neq Y_i) \leqslant \sum_i p_i^2 \leqslant \lambda q.$
- $P(X = k) - \frac{\lambda}{k!}e^{-\lambda} = P(W = k) - P(Y = k).$
- $= P(W = k, W \neq Y) - P(Y = k, W \neq Y).$
- 于是,

$$\sum_k |\star - \star| \leqslant \sum_k P(Y = k, W \neq Y) + \sum_k P(W = k, W \neq Y)$$

$$\leqslant 2P(W \neq Y) \leqslant 2\lambda q.$$